2.8 QUOTIENT RULE

Sometimes a tensor-like object unknown if it is a tensor or not; in such cases a test based on the “quotient rule” can be used to clarify the situation. According to this rule if it is not known if (K) is a tensor but it is known that (A) and (B) are tensors; the following relations holds true in all rotated coordinate frames , then A is also a tensor

$$K_i A_i = B$$

$$K_{ij} A_j = B_i$$

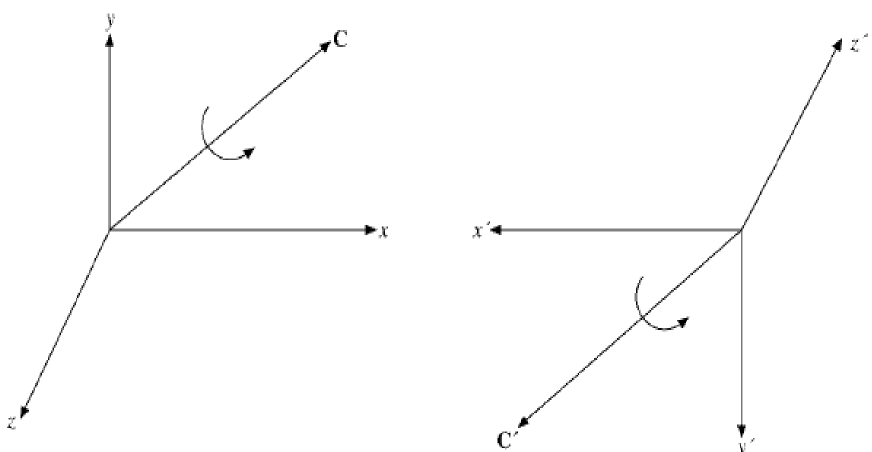
$$K_{ij} A_{jk} = B_{ik}$$

$$K_{ijkl} A_{ij} = B_{kl}$$

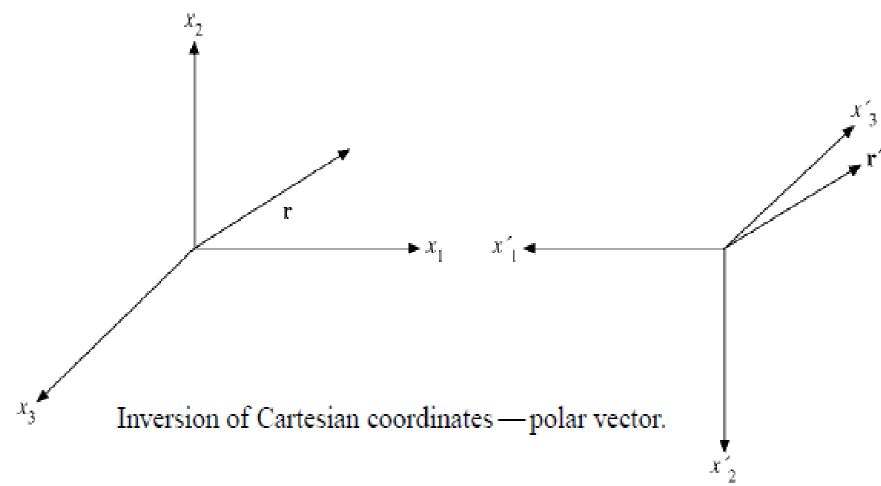
$$K_{ij} A_k = B_{ijk}.$$

2.9 PSEUDOTENSORS, DUAL TENSORS

- We have seen the transformations restricted to pure rotations. We now consider the effect of reflections or inversions.
- The tensors that are invariant under transformations involving inversion of coordinate axes through the origin**,** are called **polar(true) tensors and have odd parity.**
- While the tensors that are not invariant under transformations involving inversion of coordinate axes through the origin, are called **pseudo (axial) tensors and have even parity.**



Inversion of Cartesian coordinates—axial vector.



Inversion of Cartesian coordinates—polar vector.

•The direct product of even number of pseudo tensors is a true tensor,

while the direct product of odd number of pseudo tensors is a pseudo tensor. The direct product of true tensors is obviously a true tensor.

• The direct product of a mix of true and pseudo tensors is a true or pseudo tensor depending on the number of pseudo tensors involved in the product as being even or odd respectively.

Examples of polar vectors are displacement and acceleration, while examples of axial vectors are angular momentum and cross product of polar vectors in general

Dual Tensors

•With any **antisymmetric second-rank tensor C^{ij} (in three-dimensional space) we may associate** a dual pseudovector C_i *defined by*

$$C_i = \frac{1}{2}\varepsilon_{ijk}C^{jk}.$$

Levi-Civita symbol $\varepsilon_{ij\,k}$

$\varepsilon 123 = \varepsilon 231 = \varepsilon 312 = 1,$

$\varepsilon 132 = \varepsilon 213 = \varepsilon 321 = -1,$

all other $\varepsilon_{ij\,k} = 0.$

•The pseudovector C_i and the antisymmetric tensor **C^{ij} are** identified as **dual tensors; they are simply different representations of the same information**

2.10 GENERAL TENSORS

Metric Tensor :

Consider the transformation of vectors from one set of coordinates *(q^1, q^2, q^3)* to another **r = *(x^1, x^2, x^3).***

The new coordinates are (in general **nonlinear) functions** *x^i(q^1, q^2, q^3) of the old , so that*

$$d\mathbf{r} = \varepsilon_j dq^j, \qquad \varepsilon_i = \frac{\partial \mathbf{r}}{\partial q^i}, \qquad \varepsilon_i = h_i \mathbf{e}_i$$

The ***ε_i* *are related to the unit vectors e_i by the scale factors h_i. The e_i have no*** dimensions; the ***ε_i have the dimensions of h_i***

A **contravariant vector *V_i under general coordinate*** transformations its components transforms according to

$$V'^i = \frac{\partial x^i}{\partial q^j} V^j \qquad \text{or} \qquad \mathbf{V}' = V^j \varepsilon_j$$

the square of a differential displacement

$$(ds)^2 = d\mathbf{r} \cdot d\mathbf{r} = \left(\boldsymbol{\varepsilon}_i \, dq^i\right)^2 = \boldsymbol{\varepsilon}_i \cdot \boldsymbol{\varepsilon}_j \, dq^i \, dq^j .$$

we identify $\boldsymbol{\varepsilon}_i \cdot \boldsymbol{\varepsilon}_j$ ***as the covariant***
metric tensor $\boldsymbol{g_{ij}}$

$$\boldsymbol{\varepsilon}_i \cdot \boldsymbol{\varepsilon}_j = g_{ij} .$$

Properties of metric tensor :

The metric tensor has also a contravariant form, i.e. g^{ij}

$$g_{ij} = \mathbf{E}_i \cdot \mathbf{E}_j \qquad , \qquad g^{ij} = \mathbf{E}^i \cdot \mathbf{E}^j$$

The mixed type metric tensor is given by

$$g^i{}_j = \mathbf{E}^i \cdot \mathbf{E}_j = \delta^i{}_j \qquad , \qquad g_i{}^j = \mathbf{E}_i \cdot \mathbf{E}^j = \delta_i{}^j$$

For Cartesian coordinate systems, which are orthonormal at-space systems,

$$g^{ij} = \delta^{ij} = g_{ij} = \delta_{ij}$$

The metric tensor is symmetric,

$$g_{ij} = g_{ji} \qquad , \qquad g^{ij} = g^{ji}$$

The contravariant metric tensor is used for raising indices of covariant tensors and the covariant metric tensor is used for lowering indices of contravariant tensors,

$$A^i = g^{ij} A_j \qquad\qquad A_i = g_{ij} A^j$$

As well as a mixed form. But, the order of the indices should be respected in this process,

$$A^i{}_j = g_{jk} A^{ik} \neq A_j{}^i = g_{jk} A^{ki}$$

The covariant and contravariant metric tensors are inverses of each other,

$$[g_{ij}] = [g^{ij}]^{-1} \quad , \quad [g^{ij}] = [g_{ij}]^{-1}$$

$$g^{ik} g_{kj} = \delta^{i}_{\ j} \quad , \quad g_{ik} g^{kj} = \delta_{i}^{\ j}$$

For orthogonal coordinate systems the metric tensor is diagonal, i.e. $g_{ij} = g^{ij} = 0$ for $i \neq j$.

For flat-space orthonormal Cartesian , cylindrical and spherical coordinate systems in a 3D space, the metric tensor is given by,

$$[g_{ij}] = [\delta_{ij}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = [\delta^{ij}] = [g^{ij}]$$

$$[g_{ij}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \rho^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad [g^{ij}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\rho^2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$[g_{ij}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2 \sin^2\theta \end{bmatrix}, \quad [g^{ij}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{r^2} & 0 \\ 0 & 0 & \frac{1}{r^2 \sin^2\theta} \end{bmatrix}$$

Christoffel Symbols and Covariant Derivative

Direct differentiation of a tensor of the following form

$$V'^i = \frac{\partial x^i}{\partial q^j} V^j \quad \text{or} \quad \mathbf{V}' = V^j \varepsilon_j$$

Is given by

$$\frac{\partial \mathbf{V}'}{\partial q^j} = \frac{\partial V^i}{\partial q^j} \varepsilon_i + V^i \frac{\partial \varepsilon_i}{\partial q^j}$$

This differs from the transformation law for a second-rank mixed tensor by the second term

Now by defining the Christoffel symbol of the **second kind Γ^k_{ij}**

$$\frac{\partial \varepsilon_i}{\partial q^j} = \frac{\partial^2 \mathbf{r}}{\partial q^j \, \partial q^i} = \frac{\partial \varepsilon_j}{\partial q^i} = \Gamma^k_{ji} \varepsilon_k \quad \text{or} \quad \Gamma^m_{ij} = \varepsilon^m \cdot \frac{\partial \varepsilon_i}{\partial q^j}$$

The direct differntiation will take the form

$$\frac{\partial \mathbf{V}'}{\partial q^j} = \left(\frac{\partial V^i}{\partial q^j} + V^k \Gamma^i_{kj} \right) \varepsilon_i$$

The quantity in parenthesis is labeled a **covariant derivative,** $V^i_{;j}$

$$V^i_{;j} \equiv \frac{\partial V^i}{\partial q^j} + V^k \Gamma^i_{kj}$$

The *;j subscript indicates differentiation with respect to* q^j *.*

it is important to realize that although they bristle with indices, **neither** $\partial V^k / \partial q^j$ **nor** $\Gamma^k_{j\nu}$ have individually the correct transformation properties to be tensors. It is only the combination that has the requisite transformational attributes.

*The differential d***V** ***becomes***

$$d\mathbf{V}' = \frac{\partial \mathbf{V}'}{\partial q^j} dq^j = [V^i_{;j} dq^j] \varepsilon_i$$

the covariant derivative of a **covariant vector** *is given by*

$$V_{i;j} = \frac{\partial V_i}{\partial q^j} - V_k \Gamma^k_{ij}$$

Christoffel symbols are symmetric in the two lower indices

$$\Gamma^k_{ij} = \Gamma^k_{ji}.$$

It is often convenient to have an explicit expression for the Christoffel symbols in terms of derivatives of the metric tensor.

$$\Gamma^n_{ij} = g^{nk}[ij,k] = \frac{1}{2} g^{nk} \left[\frac{\partial g_{ik}}{\partial q^j} + \frac{\partial g_{jk}}{\partial q^i} - \frac{\partial g_{ij}}{\partial q^k} \right]$$

For Cartesian coordinate systems, the Christoel symbols are zero for all the values of indices. The covariant derivative is the same as the normal partial derivative for all tensor ranks.

The covariant derivative of the metric tensor is zero in all coordinate systems.

Several rules of normal differentiation similarly apply to covariant differentiation.

$$\partial_{;i}\left(a\mathbf{A} \pm b\mathbf{B}\right) = a\partial_{;i}\mathbf{A} \pm b\partial_{;i}\mathbf{B} \quad \text{and} \quad \partial_{;i}\left(\mathbf{AB}\right) = \left(\partial_{;i}\mathbf{A}\right)\mathbf{B} + \mathbf{A}\partial_{;i}\mathbf{B}$$

2.11 TENSOR DERIVATIVE OPERATORS

Gradient

$$\nabla\psi = \frac{\partial\psi}{\partial q^i}\boldsymbol{\epsilon}^i$$

Divergence

$$\nabla\cdot\mathbf{V} = V^i_{;i} = \frac{1}{g^{1/2}}\frac{\partial}{\partial q^k}\left(g^{1/2}V^k\right).$$

For the **orthogonal** systems note that $h_1h_2h_3 = g^{1/2}$ and $V^i = V_i/h_i$

Laplacian

$$\nabla\cdot\nabla\psi = \frac{1}{g^{1/2}}\frac{\partial}{\partial q^i}\left(g^{1/2}g^{ik}\frac{\partial\psi}{\partial q^k}\right)$$

For the **orthogonal** systems $g^{ii} = (h_i)^{-2}$

$$\nabla\cdot\nabla\psi = \frac{1}{h_1h_2h_3}\frac{\partial}{\partial q^i}\left(\frac{h_1h_2h_3}{h_i^2}\frac{\partial\psi}{\partial q^i}\right)$$

Curl

The difference of derivatives that appears in the curl has components that can be written

$$\frac{\partial V_i}{\partial q^j} - \frac{\partial V_j}{\partial q^i} = \frac{\partial V_i}{\partial q^j} - V_k \Gamma^k_{ij} - \frac{\partial V_j}{\partial q^i} + V_k \Gamma^k_{ji} = V_{i;j} - V_{j;i}$$

Exercises :

2.5.17

2.9.11