2.4 CIRCULAR CYLINDER COORDINATES

$(q1, q2, q3) \longrightarrow (\rho, \phi, z)$

$$0 \leqslant \rho < \infty, \qquad 0 \leqslant \varphi \leqslant 2\pi, \qquad \text{and} \qquad -\infty < z < \infty.$$

$$x = \rho \cos \phi, \quad y = \rho \sin \phi, \quad z = z.$$

$$\rho = \left(x^2 + y^2\right)^{1/2} \quad \varphi = \tan^{-1}\left(\frac{y}{x}\right) \qquad z = z$$

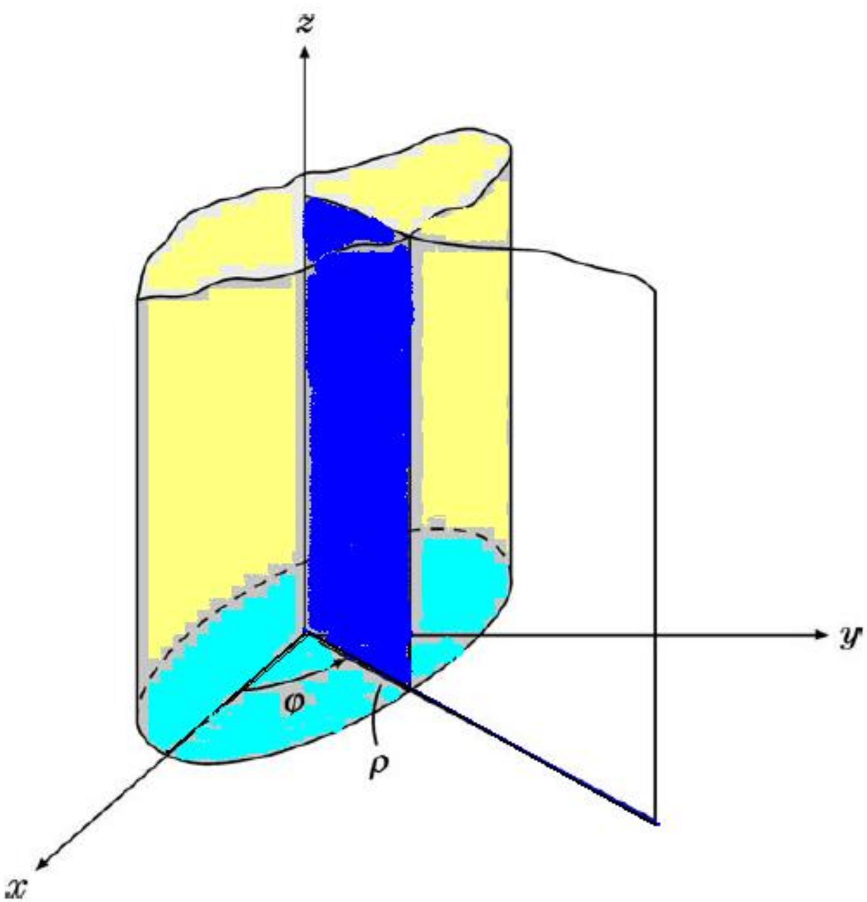
the scale factors are

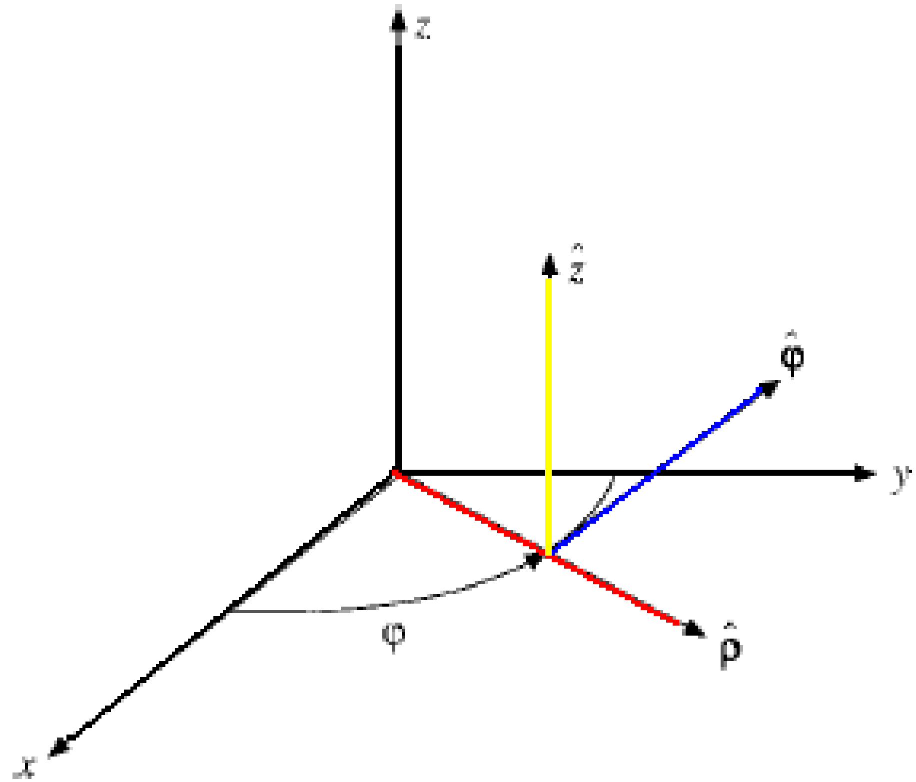
$$h_1 = h_\rho = 1,\ h_2 = h_\phi = \rho,\ h_3 = h_z = 1.$$

They are mutually orthogonal,

$$\hat{\rho} \cdot \hat{\varphi} = \hat{\varphi} \cdot \hat{\mathbf{z}} = \hat{\mathbf{z}} \cdot \hat{\rho} = 0,$$

Circular cylinder coordinates.





The coordinate vector and a general vector **V are**

$$\mathbf{r} = \hat{\boldsymbol{\rho}}\rho + \hat{\mathbf{z}}z, \qquad \mathbf{V} = \hat{\boldsymbol{\rho}}V_\rho + \hat{\boldsymbol{\varphi}}V_\varphi + \hat{\mathbf{z}}V_z.$$

A differential displacement *d**r***

$$d\mathbf{r} = \hat{\boldsymbol{\rho}}\,d\rho + \hat{\boldsymbol{\varphi}}\rho\,d\varphi + \hat{\mathbf{z}}\,dz.$$

The differential operations involving **∇**

$$\nabla\psi(\rho,\varphi,z) = \hat{\boldsymbol{\rho}}\frac{\partial\psi}{\partial\rho} + \hat{\boldsymbol{\varphi}}\frac{1}{\rho}\frac{\partial\psi}{\partial\varphi} + \hat{\mathbf{z}}\frac{\partial\psi}{\partial z},$$

$$\nabla\cdot\mathbf{V} = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho V_\rho) + \frac{1}{\rho}\frac{\partial V_\varphi}{\partial\varphi} + \frac{\partial V_z}{\partial z},$$

$$\nabla^2\psi = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\psi}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2\psi}{\partial\varphi^2} + \frac{\partial^2\psi}{\partial z^2},$$

$$\nabla\times\mathbf{V} = \frac{1}{\rho}\begin{vmatrix} \hat{\boldsymbol{\rho}} & \rho\hat{\boldsymbol{\varphi}} & \hat{\mathbf{z}} \\ \dfrac{\partial}{\partial\rho} & \dfrac{\partial}{\partial\varphi} & \dfrac{\partial}{\partial z} \\ V_\rho & \rho V_\varphi & V_z \end{vmatrix}.$$

Example 2.4.2

•The Navier–Stokes equations of hydrodynamics contain a nonlinear term

$$\nabla \times \left[\mathbf{v} \times (\nabla \times \mathbf{v})\right], \quad \text{and} \quad \mathbf{v} = \hat{\mathbf{z}} v(\rho).$$

where ***v*** is the fluid velocity. For fluid flowing through a cylindrical pipe in the *z-direction*

•The nonlinearity makes most problems difficult or impossible to solve, but in cylindrical coordinate system this tem vanishes

$$\nabla \times \mathbf{v} = \frac{1}{\rho}\begin{vmatrix} \hat{\rho} & \rho\hat{\varphi} & \hat{\mathbf{z}} \\ \dfrac{\partial}{\partial \rho} & \dfrac{\partial}{\partial \varphi} & \dfrac{\partial}{\partial z} \\ 0 & 0 & v(\rho) \end{vmatrix} = -\hat{\varphi}\frac{\partial v}{\partial \rho} \quad , \quad \mathbf{v} \times (\nabla \times \mathbf{v}) = \begin{vmatrix} \hat{\rho} & \hat{\varphi} & \hat{\mathbf{z}} \\ 0 & 0 & v \\ 0 & -\dfrac{\partial v}{\partial \rho} & 0 \end{vmatrix} = \hat{\rho}\, v(\rho)\frac{\partial v}{\partial \rho}.$$

$$\nabla \times \big(\mathbf{v} \times (\nabla \times \mathbf{v})\big) = \frac{1}{\rho}\begin{vmatrix} \hat{\rho} & \rho\hat{\varphi} & \hat{\mathbf{z}} \\ \dfrac{\partial}{\partial \rho} & \dfrac{\partial}{\partial \varphi} & \dfrac{\partial}{\partial z} \\ v\dfrac{\partial v}{\partial \rho} & 0 & 0 \end{vmatrix} = 0,$$

2.5 SPHERICAL POLAR COORDINATES

$(q1, q2, q3) \Rightarrow (r, \vartheta, \phi)$

$$0 \leqslant r < \infty,\ 0 \leqslant \theta \leqslant \pi,\ \text{and}\ 0 \leqslant \varphi \leqslant 2\pi$$

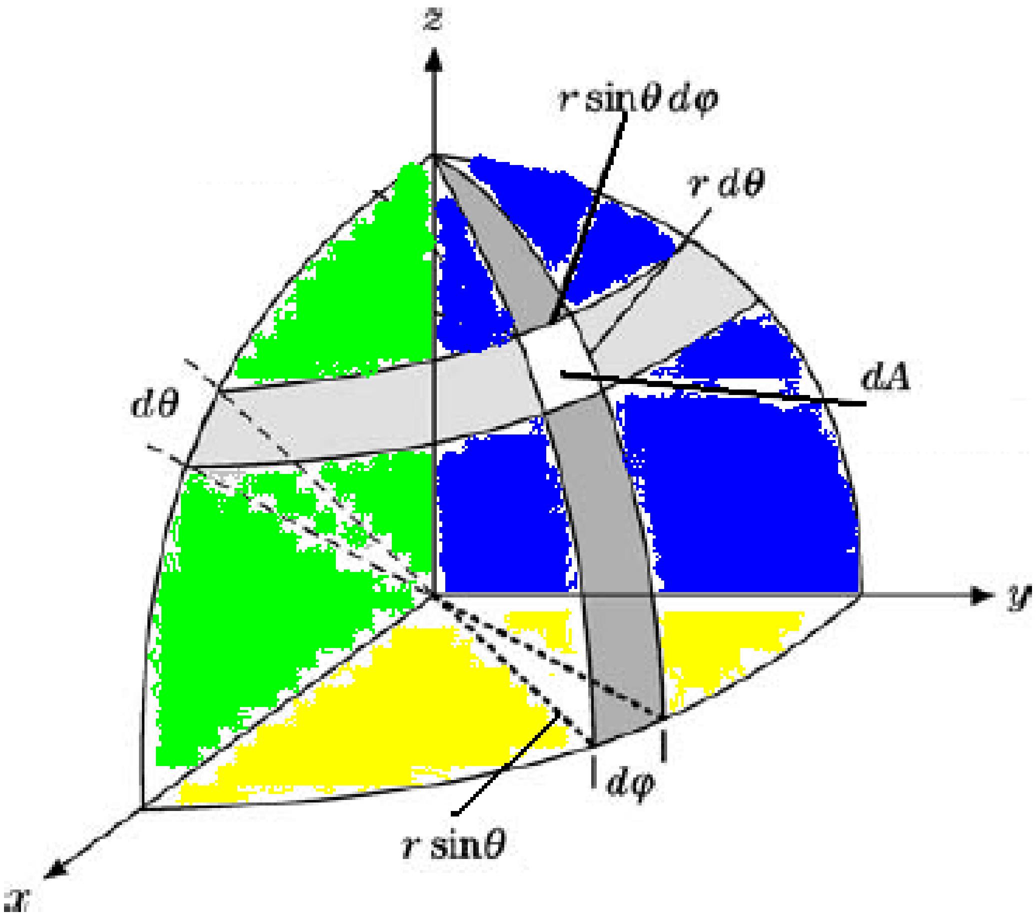
$$x = r\sin\theta\cos\varphi, \qquad y = r\sin\theta\sin\varphi, \qquad z = r\cos\theta$$

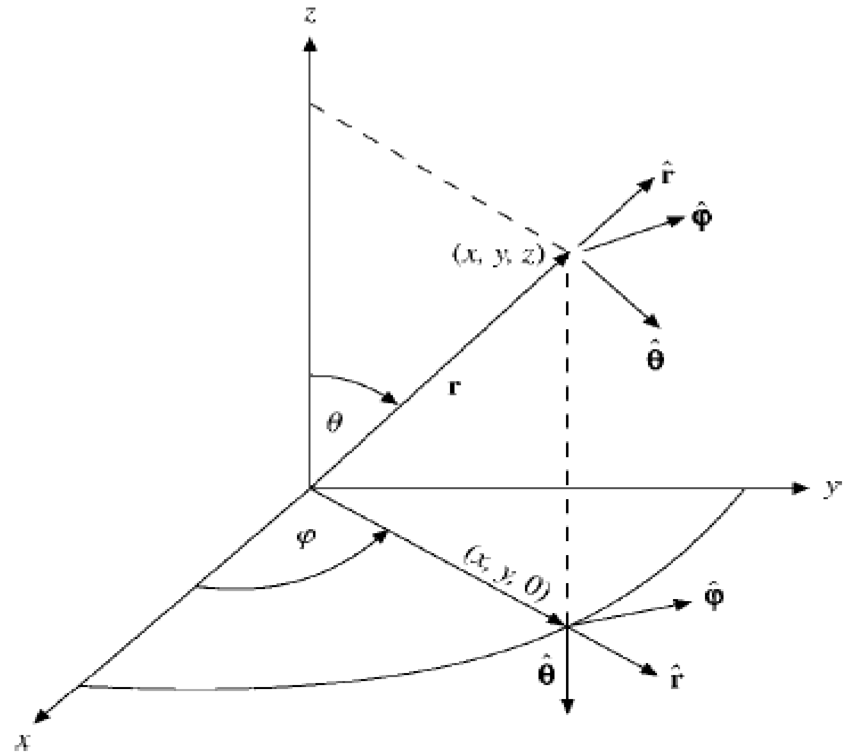
$$r = \sqrt{x^2 + y^2 + z^2}\ ;\quad \phi = \tan^{-1}\left(\frac{y}{x}\right)\ ;\quad \theta = \cos^{-1}\left(\frac{z}{r}\right) = \cos^{-1}\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right)$$

the scale factors are

$$h_1 = h_r = 1, \quad h_2 = h_\theta = r, \quad h_3 = h_\varphi = r\sin\theta$$

Spherical polar coordinate area





•A differential displacement dr

$$d\mathbf{r} = \hat{\mathbf{r}}\, dr + \hat{\theta}\, r\, d\theta + \hat{\varphi}\, r \sin\theta\, d\varphi$$

•the coordinates are orthogonal

$$ds^2 = d\mathbf{r} \cdot d\mathbf{r} = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta\, d\varphi^2$$

•None of the Carteisan unit vectors are dependent on position, but this is not true for non-Cartesian unit vectors . the unit vectors ˆr, ˆθ , and ˆϕ vary in direction as the angles θ and ϕ vary

$$\hat{\mathbf{r}} = \hat{\mathbf{x}} \sin\theta \cos\varphi + \hat{\mathbf{y}} \sin\theta \sin\varphi + \hat{\mathbf{z}} \cos\theta,$$

$$\hat{\theta} = \hat{\mathbf{x}} \cos\theta \cos\varphi + \hat{\mathbf{y}} \cos\theta \sin\varphi - \hat{\mathbf{z}} \sin\theta = \frac{\partial \hat{\mathbf{r}}}{\partial \theta},$$

$$\hat{\varphi} = -\hat{\mathbf{x}} \sin\varphi + \hat{\mathbf{y}} \cos\varphi = \frac{1}{\sin\theta} \frac{\partial \hat{\mathbf{r}}}{\partial \varphi},$$

•This set of equation can be proved geometrically from the previous shape or algebraically by eq. 2.9 in textbook

•When integrating vector function in Cartesian coordinates we can take the unit vectors outside the integral, since they are constant. This is no longer true in non Cartesian coordinates. Often it's better to convert your unit vectors back to Cartesian before attempting to do any integration.

The position vector r may be written as

$$\begin{aligned}\mathbf{r} = \hat{\mathbf{r}}r &= \hat{\mathbf{x}}x + \hat{\mathbf{y}}y + \hat{\mathbf{z}}z \\ &= \hat{\mathbf{x}}r\sin\theta\cos\varphi + \hat{\mathbf{y}}r\sin\theta\sin\varphi + \hat{\mathbf{z}}r\cos\theta.\end{aligned}$$

the area element (for *r = constant) is*

$$dA = r^2 \sin\theta\, d\theta\, d\varphi, \qquad d\Omega = \frac{dA}{r^2} = \sin\theta\, d\theta\, d\varphi.$$

the volume element is

$$d\tau = r^2\, dr \sin\theta\, d\theta\, d\varphi = r^2\, dr\, d\Omega.$$

The differential operations involving ∇

$$\nabla\psi = \hat{\mathbf{r}}\frac{\partial\psi}{\partial r} + \hat{\boldsymbol{\theta}}\frac{1}{r}\frac{\partial\psi}{\partial\theta} + \hat{\boldsymbol{\varphi}}\frac{1}{r\sin\theta}\frac{\partial\psi}{\partial\varphi},$$

$$\nabla\cdot\mathbf{V} = \frac{1}{r^2\sin\theta}\left[\sin\theta\frac{\partial}{\partial r}(r^2V_r) + r\frac{\partial}{\partial\theta}(\sin\theta V_\theta) + r\frac{\partial V_\varphi}{\partial\varphi}\right],$$

$$\nabla\cdot\nabla\psi = \frac{1}{r^2\sin\theta}\left[\sin\theta\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{\sin\theta}\frac{\partial^2\psi}{\partial\varphi^2}\right],$$

$$\nabla\times\mathbf{V} = \frac{1}{r^2\sin\theta}\begin{vmatrix} \hat{\mathbf{r}} & r\hat{\boldsymbol{\theta}} & r\sin\theta\hat{\boldsymbol{\varphi}} \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial\theta} & \dfrac{\partial}{\partial\varphi} \\ V_r & rV_\theta & r\sin\theta V_\varphi \end{vmatrix}.$$

Example 2.5.1 ∇, ∇· , ∇× FOR A CENTRAL FORCE

- Using Gradient in polar coordinates

$$\nabla f(r) = \hat{\mathbf{r}} \frac{df}{dr},$$
$$\nabla r^n = \hat{\mathbf{r}} n r^{n-1}.$$

For the Coulomb potential $V = Ze/(4\pi\varepsilon_0 r)$, the electric field is $\mathbf{E} = -\nabla V = \frac{Ze}{4\pi\varepsilon_0 r^2}\hat{\mathbf{r}}$.

- Using Divergence in polar coordinates

$$\nabla \cdot \hat{\mathbf{r}} f(r) = \frac{2}{r} f(r) + \frac{df}{dr},$$
$$\nabla \cdot \hat{\mathbf{r}} r^n = (n+2) r^{n-1}.$$

For $r > 0$ the charge density of the electric field of the Coulomb potential is

$\rho = \nabla \cdot \mathbf{E} = \frac{Ze}{4\pi\varepsilon_0} \nabla \cdot \frac{\hat{\mathbf{r}}}{r^2} = 0$ because $n = -2$.

- Using Curl in polar coordinates $\nabla \times \hat{\mathbf{r}} f(r) = 0.$
- Laplasian will be

$$\nabla^2 f(r) = \frac{2}{r}\frac{df}{dr} + \frac{d^2 f}{dr^2},$$
$$\nabla^2 r^n = n(n+1) r^{n-2},$$

2.6 TENSOR ANALYSIS

Introduction, Definitions

- A tensor of rank n has 3^n components that transform in a definite way
- Scalars and vectors are special cases of tensors
- A scalar is specified by one real number and is a tensor of rank 0.
- In three-dimensional space, a vector is specified by $3 = 3^1$ real numbers, its Cartesian components, and is a tensor of rank 1.
- In N-dimensional space a tensor of rank n has N^n components.

•This transformation philosophy is of central importance for tensor analysis.

•Any set of quantities A^j transforming according to

$$A'^i = \sum_j \frac{\partial x'_i}{\partial x_j} A^j$$

is defined as a contravariant vector, (displacement, velocity, accelaration,….)

•Any set of quantities A_j transforming according to

$$A'_i = \sum_j \frac{\partial x^j}{\partial x'^i} A_j$$

as the definition of a covariant vector, (e.g. gradient of scalar)

•The components of any contravariant vector are denoted by a superscript, A^i , whereas a subscript is used for the components of a covariant vector A_i

•Only in Cartesian coordinates is

$$\frac{\partial x^j}{\partial x'^i} = \frac{\partial x'^i}{\partial x^j}$$

so that there no difference between contravariant and covariant transformations.

Definition of Tensors of Rank 2

$$A'^{ij} = \sum_{kl} \frac{\partial x'^i}{\partial x^k} \frac{\partial x'^j}{\partial x^l} A^{kl}, \qquad \textbf{contravariant}$$

$$B'^i{}_j = \sum_{kl} \frac{\partial x'^i}{\partial x^k} \frac{\partial x^l}{\partial x'^j} B^k{}_l, \qquad \textbf{mixed,}$$

$$C'_{ij} = \sum_{kl} \frac{\partial x^k}{\partial x'^i} \frac{\partial x^l}{\partial x'^j} C_{kl}. \qquad \textbf{covariant}$$

•Clearly, the rank goes as the number of partial derivatives

•The number of indices (equal to the rank of tensor) is independent of the dimensions of the space

•In Cartesian coordinates, all three forms of the tensors of second rank contravariant, mixed, and covariant are—the same.

Addition and Subtraction of Tensors

The addition and subtraction of tensors is defined in terms of the individual elements

$$\mathbf{A} + \mathbf{B} = \mathbf{C},$$

$$A^{ij} + B^{ij} = C^{ij}.$$

A and B must be tensors of the same rank and both expressed in a space of the same number of dimensions.

Summation Convention

- Let us agree that when an index appears on one side of an equation, once as a superscript and once as a subscript (except for the coordinates where both are subscripts), we automatically sum over that index

$$B'^{i}{}_{j} = \frac{\partial x'^{i}}{\partial x^{k}} \frac{\partial x^{l}}{\partial x'^{j}} B^{k}{}_{l}$$

- With the summation of the right-hand side over k and l implied

Symmetry–Antisymmetry

In general, A^{mn} is independent of A^{nm}, but If, for all m and n,

$$A^{mn} = A^{nm}$$

we call the tensor symmetric

And If,

$$A^{mn} = -A^{nm}$$

the tensor is antisymmetric

Contraction

•In contraction, two indices, one covariant and the other contravariant, are set equal to each other

•For example,let us contract the second-rank mixed tensor $B'^{i}{}_{j}$

$$B'^{i}{}_{i} = \frac{\partial x'^{i}}{\partial x^{k}} \frac{\partial x^{l}}{\partial x'^{i}} B^{k}{}_{l} = \frac{\partial x^{l}}{\partial x^{k}} B^{k}{}_{l}$$

$$B'^{i}{}_{i} = \delta^{l}{}_{k} B^{k}{}_{l} = B^{k}{}_{k}.$$

•In general, the operation of contraction reduces the rank of a tensor by 2.

Direct Product

In general, the direct product of two tensors is a tensor of rank equal to the sum of the two initial ranks

The components of a covariant vector (first-rank tensor) a_i and those of a contravariant vector (first-rank tensor) b^j may be multiplied component by component to give a second-rank tensor

$$a'_i b'^j = \frac{\partial x^k}{\partial x'^i} a_k \frac{\partial x'^j}{\partial x^l} b^l = \frac{\partial x^k}{\partial x'^i} \frac{\partial x'^j}{\partial x^l} (a_k b^l).$$

And

$$A^i{}_j B^{kl} = C^i{}_j{}^{kl},$$

where $C^i{}_j{}^{kl}$ is a tensor of fourth rank.

$$C'^i{}_j{}^{kl} = \frac{\partial x'^i}{\partial x^m} \frac{\partial x^n}{\partial x'^j} \frac{\partial x'^k}{\partial x^p} \frac{\partial x'^l}{\partial x^q} C^m{}_n{}^{pq}.$$

The direct product is a technique for creating new, higher-rank tensors