1.14 GAUSS' LAW, POISSON'S EQUATION

Gauss' Law

The electric field **E of** point charge *q at the origin of our coordinate system* given by

$$\mathbf{E} = \frac{q\hat{\mathbf{r}}}{4\pi\varepsilon_0 r^2}.$$

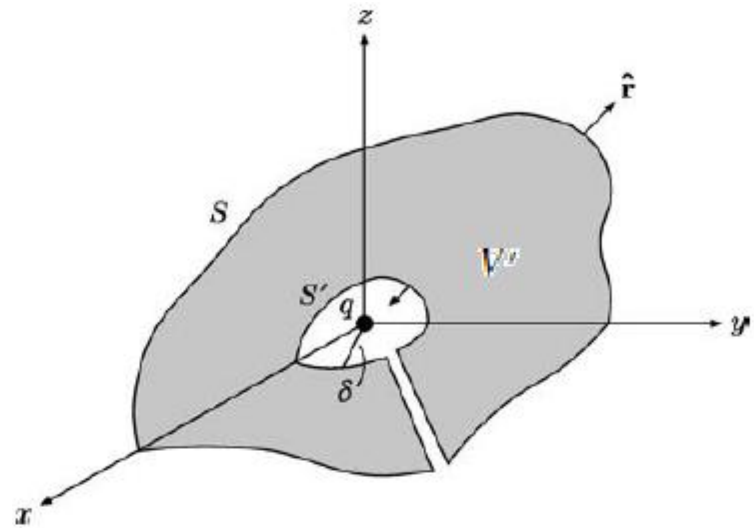
1- If we have a surface (*S = ∂V)*does not include the origin, it is easily to prove that (see Example 1.7.2)

$$\int_V \nabla \cdot \mathbf{E} = 0 \xrightarrow{\text{Gauss' theorem}} \oint_{\partial V} \mathbf{E} \cdot d\sigma = 0 \qquad \text{if } \partial V \text{ does not enclose } q.$$

2- If *q is within the volume V,* Gauss' theorem cab be applied to region *V' which include three part*

- surface *S'* ,surround **r = 0 by a small** spherical hole (of radius δ)
- boundary of *V*
- and connect the hole with the boundary of *V via a small tube*

The contribution from the connecting tube will become negligible in the limit that it shrinks toward zero cross section, as **E is finite everywhere on the** tube's surface.



The integral *will thus be* over *S plus S′*. But note that the "outward" direction for *S′ is toward smaller r , so* $d\sigma' = -\hat{\mathbf{r}}\, dA.$

Because the modified volume contains no charge, we have

$$\oint_{\partial V'} \mathbf{E} \cdot d\sigma = \oint_{S} \mathbf{E} \cdot d\sigma + \frac{q}{4\pi\varepsilon_0} \oint_{S'} \frac{\hat{\mathbf{r}} \cdot d\sigma'}{\delta^2} = 0, \qquad (1)$$

Writing *dΩ as the element of solid angle*

$$\oint_{S'} \frac{\hat{\mathbf{r}} \cdot d\sigma'}{\delta^2} = \int \frac{\hat{\mathbf{r}}}{\delta^2} \cdot (-\hat{\mathbf{r}}\,\delta^2\, d\Omega) = -\int d\Omega = -4\pi,$$

Rearranging eq. (1) ,

$$\oint_S \mathbf{E} \cdot d\sigma = -\frac{q}{4\pi\varepsilon_0}(-4\pi) = +\frac{q}{\varepsilon_0},$$

In that case, *q can be replaced by* $\int_V \rho\, d\tau$

$$\int_{\partial V} \mathbf{E} \cdot d\sigma = \int_V \frac{\rho}{\varepsilon_0}\, d\tau.$$

If we apply Gauss' theorem

$$\int_V \nabla \cdot \mathbf{E}\, d\tau = \int_V \frac{\rho}{\varepsilon_0}\, d\tau.$$

Since our volume is completely arbitrary, then

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}.$$

Poisson’s Equation

From Gauss’ law

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}.$$

If we replace **E by –∇*ϕ*,** assuming a situation independent of time,

$$\nabla \cdot \nabla\varphi = -\frac{\rho}{\varepsilon_0},$$

which is Poisson’s equation. For the condition *ρ = 0 this reduces to an even more famous* equation,

$$\nabla \cdot \nabla\varphi = 0,$$

1.16 HELMHOLTZ'S THEOREM

A vector **V satisfying**

$$\nabla \cdot \mathbf{V}_1 = s, \qquad \nabla \times \mathbf{V}_1 = \mathbf{c},$$

with both source and circulation densities vanishing at infinity may be written as the sum of two parts, one of which is irrotational, the other of which is solenoidal

$$\mathbf{V} = -\nabla\varphi + \nabla \times \mathbf{A},$$

Where,

$$\varphi(\mathbf{r}_1) = \frac{1}{4\pi}\int \frac{s(\mathbf{r}_2)}{r_{12}} d\tau_2, \qquad \mathbf{A}(\mathbf{r}_1) = \frac{1}{4\pi}\int \frac{\mathbf{c}(\mathbf{r}_2)}{r_{12}} d\tau_2.$$

Here the argument $\mathbf{r}_1$ **indicates** ***the field point;*** $\mathbf{r}_2$***, the coordinates of the*** source point

- If *s = 0, then* ***V is solenoidal*** (i.e. *divergence-less*) ***and that implies*** $\phi = 0$***.*** ***and V =*** **∇ × A,**
- If **c = 0, then V is irrotational (i.e.** curl-less) **and that implies A = 0, and V=−∇**ϕ**,**

Additional part

A. Example in electrostatics

We have

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}, \tag{18a}$$

$$\nabla \times \mathbf{E} = 0, \tag{18b}$$

where $\mathbf{E}$ is the electric field and ρ the charge density. Hence $\boxed{\mathbf{E} = -\nabla V,}$ where the scalar potential V is given by the familiar expression

$$\boxed{V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \, d\tau',} \tag{19}$$

as discussed in the course.

B. Example in magnetistatics

We have

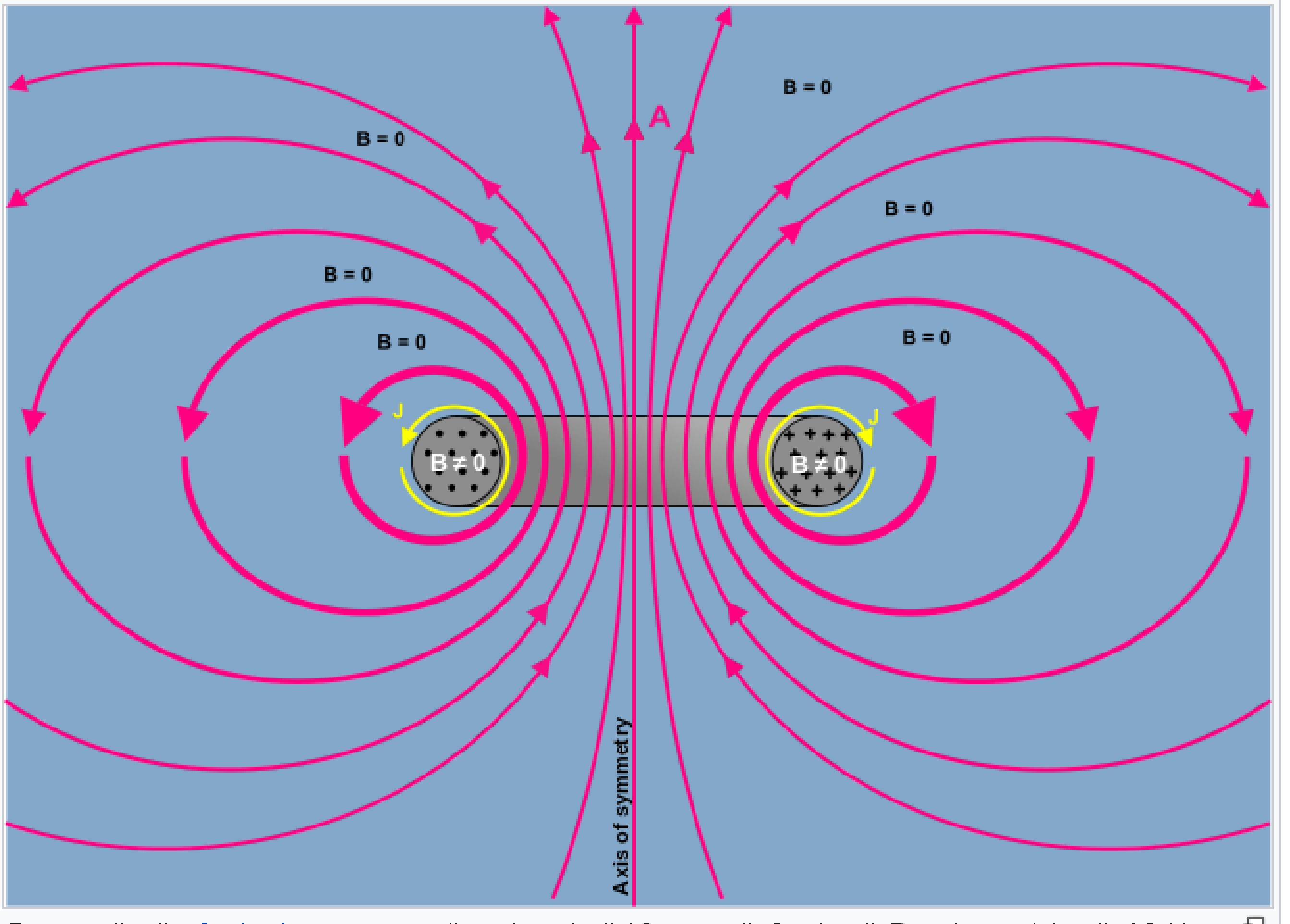
$$\nabla \cdot \mathbf{B} = 0, \tag{20a}$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}, \tag{20b}$$

where $\mathbf{B}$ is the magnetic field and $\mathbf{J}$ the current density. Hence $\boxed{\mathbf{B} = \nabla \times \mathbf{A},}$ where the vector potential $\mathbf{A}$ is given, in the gauge with $\nabla \cdot \mathbf{A} = 0$, by

$$\boxed{\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \, d\tau',} \tag{21}$$

Additional part



Representing the Coulomb gauge magnetic vector potential **A**, magnetic flux density **B**, and current density **J** fields around a toroidal inductor of circular cross section. Thicker lines indicate field lines of higher average intensity. Circles in the cross section of the core represent the **B**-field coming out of the picture, plus signs represent **B**-field going into the picture. $\nabla \cdot \mathbf{A} = 0$ has been assumed.

CHAPTER 2

VECTOR ANALYSIS IN CURVED COORDINATES AND TENSORS

2.1 ORTHOGONAL COORDINATES IN R3

Not all physical problems are well adapted to a solution in Cartesian coordinates. For example, the Schrödinger equation for the hydrogen atom is best solved using spherical polar coordinates.

The point is that the coordinate system should be chosen to fit the problem, to exploit any constraint or symmetry present in it.

We only look at *orthogonal* coordinate systems, so that locally the three axes (such as r,θ,ϕ) are a mutually perpendicular set.

We may describe any point *(x, y, z) as the intersection* of three planes in Cartesian coordinates or as the intersection of the three surfaces that form our new, curvilinear coordinates.

we may identify our point by (q1, q2, q3) as well as by (x, y, z):

General curvilinear coordinates	Circular cylindrical coordinates
q_1, q_2, q_3	ρ, φ, z
$x = x(q_1, q_2, q_3)$	$-\infty < x = \rho\cos\varphi < \infty$
$y = y(q_1, q_2, q_3)$	$-\infty < y = \rho\sin\varphi < \infty$
$z = z(q_1, q_2, q_3)$	$-\infty < z = z < \infty$

specifying x, y, z in terms of q_1, q_2, q_3 and the inverse relations

$q_1 = q_1(x, y, z)$	$0 \leqslant \rho = \left(x^2 + y^2\right)^{1/2} < \infty$
$q_2 = q_2(x, y, z)$	$0 \leqslant \varphi = \arctan(y/x) < 2\pi$
$q_3 = q_3(x, y, z)$	$-\infty < z = z < \infty.$

we can associate a unit vector $\hat{q}_i$ normal to the surface q_i = constant and in the direction of increasing q_i . In general, these unit vectors will depend on the position in space. Then a vector V may be written

$$\mathbf{V} = \hat{\mathbf{q}}_1 V_1 + \hat{\mathbf{q}}_2 V_2 + \hat{\mathbf{q}}_3 V_3,$$

Differentiation of ***r*** in vector notation

$$d\mathbf{r} = \sum_i \frac{\partial \mathbf{r}}{\partial q_i} dq_i$$

square of the distance element can be written as

$$ds^2 = d\mathbf{r} \cdot d\mathbf{r} = d\mathbf{r}^2 = \sum_{ij} \frac{\partial \mathbf{r}}{\partial q_i} \cdot \frac{\partial \mathbf{r}}{\partial q_j} dq_i \, dq_j$$

$$= g_{11}\, dq_1^2 + g_{12}\, dq_1\, dq_2 + g_{13}\, dq_1\, dq_3$$

$$+ g_{21}\, dq_2\, dq_1 + g_{22}\, dq_2^2 + g_{23}\, dq_2\, dq_3$$

$$+ g_{31}\, dq_3\, dq_1 + g_{32}\, dq_3\, dq_2 + g_{33}\, dq_3^2$$

$$= \sum_{ij} g_{ij}\, dq_i\, dq_j,$$

g_{ij} *may be viewed as specifying* the nature of the coordinate system. *Collectively these coefficients are* referred to as the **metric**

•At usual we limit ourselves to orthogonal (mutually perpendicular surfaces) coordinate Systems

$$g_{ij} = 0, \quad i \neq j, \quad \text{and } \hat{\mathbf{q}}_i \cdot \hat{\mathbf{q}}_j = \delta_{ij}. \qquad \text{and} \quad g_{ii} = h_i^2 > 0,$$

So that ,

$$ds^2 = (h_1\, dq_1)^2 + (h_2\, dq_2)^2 + (h_3\, dq_3)^2 = \sum_i (h_i\, dq_i)^2.$$

scale factors *h1, h2, and h3. may be* conveniently identified by the relation, so the **product *h_i dq_i* *must*** have a dimension of length.

$$ds_i = h_i\, dq_i, \qquad \frac{\partial \mathbf{r}}{\partial q_i} = h_i \hat{\mathbf{q}}_i$$

Here, *ds_i is a differential length along the* direction $\hat{\mathbf{q}}_i$

The differential distance vector *d**r may be written as***

$$d\mathbf{r} = h_1\, dq_1\, \hat{\mathbf{q}}_1 + h_2\, dq_2\, \hat{\mathbf{q}}_2 + h_3\, dq_3\, \hat{\mathbf{q}}_3 = \sum_i h_i\, dq_i\, \hat{\mathbf{q}}_i .$$

a line integral becomes

$$\int \mathbf{V} \cdot d\mathbf{r} = \sum_i \int V_i h_i\, dq_i .$$

and the area and volume elements

$$\begin{aligned} d\sigma &= ds_2\, ds_3\, \hat{\mathbf{q}}_1 + ds_3\, ds_1\, \hat{\mathbf{q}}_2 + ds_1\, ds_2\, \hat{\mathbf{q}}_3 \\ &= h_2 h_3\, dq_2\, dq_3\, \hat{\mathbf{q}}_1 + h_3 h_1\, dq_3\, dq_1\, \hat{\mathbf{q}}_2 \\ &\quad + h_1 h_2\, dq_1\, dq_2\, \hat{\mathbf{q}}_3 . \end{aligned}$$

$$d\tau = ds_1\, ds_2\, ds_3 = h_1 h_2 h_3\, dq_1\, dq_2\, dq_3 .$$

A surface integral becomes

$$\begin{aligned} \int \mathbf{V} \cdot d\boldsymbol{\sigma} &= \int V_1 h_2 h_3\, dq_2\, dq_3 + \int V_2 h_3 h_1\, dq_3\, dq_1 \\ &\quad + \int V_3 h_1 h_2\, dq_1\, dq_2 . \end{aligned}$$

The vector **algebra is the same in orthogonal curvilinear** coordinates as in Cartesian coordinates

For example the dot product,

$$\mathbf{A} \cdot \mathbf{B} = \sum_{ik} A_i \hat{\mathbf{q}}_i \cdot \hat{\mathbf{q}}_k B_k = \sum_{ik} A_i B_k \delta_{ik}$$

$$= \sum_i A_i B_i = A_1 B_1 + A_2 B_2 + A_3 B_3,$$

For the cross product,

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \hat{\mathbf{q}}_1 & \hat{\mathbf{q}}_2 & \hat{\mathbf{q}}_3 \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{vmatrix},$$

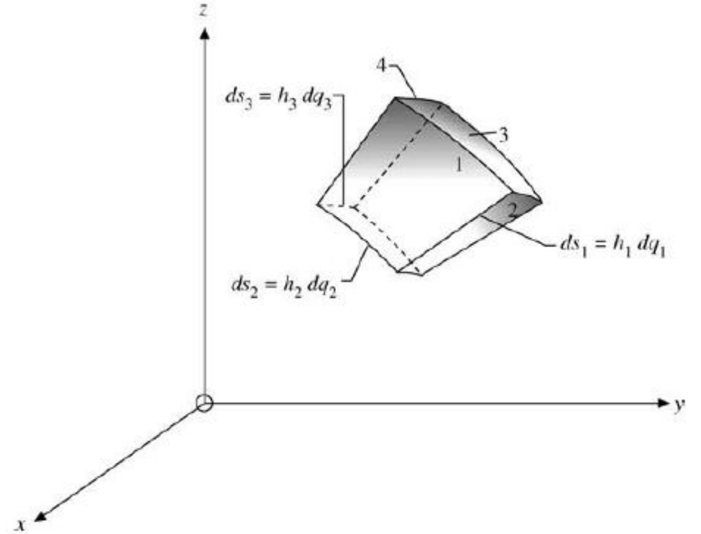
2.2 DIFFERENTIAL VECTOR OPERATORS

Gradient

Because our curvilinear coordinates are orthogonal, the gradient takes the same form as for Cartesian coordinates, providing we use the differential displacements

$$\begin{aligned}\nabla\psi(q_1,q_2,q_3) &= \hat{\mathbf{q}}_1\frac{\partial\psi}{\partial s_1}+\hat{\mathbf{q}}_2\frac{\partial\psi}{\partial s_2}+\hat{\mathbf{q}}_3\frac{\partial\psi}{\partial s_3}\\ &= \hat{\mathbf{q}}_1\frac{1}{h_1}\frac{\partial\psi}{\partial q_1}+\hat{\mathbf{q}}_2\frac{1}{h_2}\frac{\partial\psi}{\partial q_2}+\hat{\mathbf{q}}_3\frac{1}{h_3}\frac{\partial\psi}{\partial q_3}\\ &= \sum_i \hat{\mathbf{q}}_i\frac{1}{h_i}\frac{\partial\psi}{\partial q_i}.\end{aligned}$$

Divergence



This operator must have the same meaning as in Cartesian coordinates, so

$\nabla \cdot \mathbf{V}$ **must give the net outward flux of V per unit volume at the point of evaluation**

$$\text{Net } q_1 \text{ outflow} = \left[V_1 h_2 h_3 + \frac{\partial}{\partial q_1}(V_1 h_2 h_3)\, dq_1\right] dq_2\, dq_3 - V_1 h_2 h_3\, dq_2\, dq_3$$

$$= \frac{\partial}{\partial q_1}(V_1 h_2 h_3)\, dq_1\, dq_2\, dq_3,$$

Combining this with the *q_2 and q_3 outflows and dividing by the differential volume $h_1 h_2 h_3\, dq_1 dq_2 dq_3$, we get*

$$\nabla \cdot \mathbf{V}(q_1, q_2, q_3) = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial q_1}(V_1 h_2 h_3) + \frac{\partial}{\partial q_2}(V_2 h_3 h_1) + \frac{\partial}{\partial q_3}(V_3 h_1 h_2)\right].$$

From the formulas for the gradient and divergence, we can form the Laplacian in curvilinear coordinates $\mathbf{V} = \nabla\psi(q_1, q_2, q_3)$

$$\nabla \cdot \nabla\psi(q_1, q_2, q_3) = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial q_1}\left(\frac{h_2 h_3}{h_1}\frac{\partial \psi}{\partial q_1}\right) + \frac{\partial}{\partial q_2}\left(\frac{h_3 h_1}{h_2}\frac{\partial \psi}{\partial q_2}\right) + \frac{\partial}{\partial q_3}\left(\frac{h_1 h_2}{h_3}\frac{\partial \psi}{\partial q_3}\right)\right]$$

Curl

apply Stokes' theorem, we can find

$$\nabla \times \mathbf{V} = \frac{1}{h_1 h_2 h_3} \begin{vmatrix} \hat{\mathbf{q}}_1 h_1 & \hat{\mathbf{q}}_2 h_2 & \hat{\mathbf{q}}_3 h_3 \\ \dfrac{\partial}{\partial q_1} & \dfrac{\partial}{\partial q_2} & \dfrac{\partial}{\partial q_3} \\ h_1 V_1 & h_2 V_2 & h_3 V_3 \end{vmatrix}$$

Remember that, because of the presence of the differential operators, this determinant must be expanded from the top down. Note that this equation is **not identical with the form for** the cross product of two vectors. **∇ is not an ordinary vector; it is a vector operator**

Exercises

1.13.4

1.13.9

1.14.4

2.1.2

2.2.2

2.4.5

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