# 1.9 SUCCESSIVE APPLICATIONS OF ∇

Letting ∇ **operate on** gradient, divergence, and curl to obtain expressions involving second derivatives

$$
\text{(a)}\quad \nabla\cdot\nabla\varphi = \left(\hat{\mathbf{x}}\frac{\partial}{\partial x}+\hat{\mathbf{y}}\frac{\partial}{\partial y}+\hat{\mathbf{z}}\frac{\partial}{\partial z}\right)\cdot\left(\hat{\mathbf{x}}\frac{\partial\varphi}{\partial x}+\hat{\mathbf{y}}\frac{\partial\varphi}{\partial y}+\hat{\mathbf{z}}\frac{\partial\varphi}{\partial z}\right)
$$

$$
=\frac{\partial^2\varphi}{\partial x^2}+\frac{\partial^2\varphi}{\partial y^2}+\frac{\partial^2\varphi}{\partial z^2}.
$$

$\boldsymbol{\nabla\cdot\nabla\phi = 0}$ When $\phi$ *is the electrostatic potential ( see Ex. 1.9.1)*

zero here is a consequence of physics.

$$
\text{(b)}\quad \nabla\times\nabla\varphi = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial\varphi}{\partial x} & \frac{\partial\varphi}{\partial y} & \frac{\partial\varphi}{\partial z} \end{vmatrix} = \hat{\mathbf{x}}\left(\frac{\partial^2\varphi}{\partial y\,\partial z}-\frac{\partial^2\varphi}{\partial z\,\partial y}\right)+\hat{\mathbf{y}}\left(\frac{\partial^2\varphi}{\partial z\,\partial x}-\frac{\partial^2\varphi}{\partial x\,\partial z}\right)
$$

$$
+\hat{\mathbf{z}}\left(\frac{\partial^2\varphi}{\partial x\,\partial y}-\frac{\partial^2\varphi}{\partial y\,\partial x}\right)=0,
$$

$\phi$ *are continuous functions* , the zero in comes as a mathematical identity

$$\text{(c)}\quad \nabla\cdot\nabla\times\mathbf{V}=\begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_x & V_y & V_z \end{vmatrix}=0.$$

The divergence of a curl vanishes or all curls are solenoidal.

$$\text{(d)}\qquad \nabla\times(\nabla\times\mathbf{V})=\nabla\nabla\cdot\mathbf{V}-\nabla\cdot\nabla\mathbf{V},$$

This follows immediately from the *BAC–CAB rule and* valid in Cartesian coordinates (but not in curved coordinates).

## *Example 1.9.2 ELECTROMAGNETIC WAVE EQUATION*

In vacuum Maxwell's equations become

$$\nabla \cdot \mathbf{B} = 0, \quad (1.86a) \qquad \nabla \times \mathbf{B} = \varepsilon_0 \mu_0 \frac{\partial \mathbf{E}}{\partial t}, \quad (1.86c)$$

$$\nabla \cdot \mathbf{E} = 0, \quad (1.86b) \qquad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}. \quad (1.86d)$$

Suppose we eliminate **B from Eqs. (1.86c) and (1.86d). We may do this by taking the** curl of both sides of Eq. (1.86d) and the time derivative of both sides of Eq. (1.86c). Since the space and time derivatives commute,

$$\frac{\partial}{\partial t} \nabla \times \mathbf{B} = \nabla \times \frac{\partial \mathbf{B}}{\partial t}, \longrightarrow \nabla \times (\nabla \times \mathbf{E}) = -\varepsilon_0 \mu_0 \frac{\partial^2 \mathbf{E}}{\partial t^2}.$$

Application of Eqs.

$$\nabla \times (\nabla \times \mathbf{V}) = \nabla \nabla \cdot \mathbf{V} - \nabla \cdot \nabla \mathbf{V}, \quad \text{and} \quad \nabla \cdot \mathbf{E} = 0,$$

Yields the electromagnetic vector wave equation

$$\nabla \cdot \nabla \mathbf{E} = \varepsilon_0 \mu_0 \frac{\partial^2 \mathbf{E}}{\partial t^2},$$

# 1.10 VECTOR INTEGRATION

## Line Integrals

For *dr = ˆxdx+ˆy dy+ˆz dz the line integrals takes the following form*

$$\int_C \varphi \, d\mathbf{r}, \qquad \int_C \mathbf{V} \cdot d\mathbf{r}, \qquad \int_C \mathbf{V} \times d\mathbf{r},$$

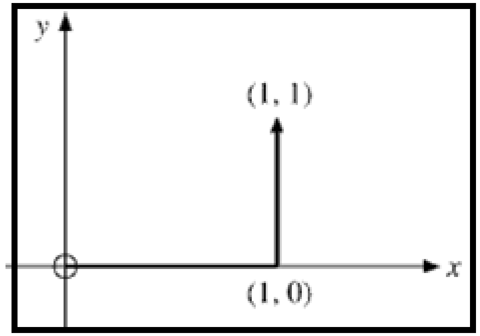
- The three integrals are Riemann integrals and the integral is over some contour *C that may be open* or closed .
- The path of integration *C must be specified. Unless the integrand has special properties so that* the integral depends only on the value of the end points
- The method of approach will be to reduce the vector integral to scalar integrals .
- With $\phi$*, a scalar, the first integral reduces to*

$$\int_C \varphi \, d\mathbf{r} = \hat{\mathbf{x}} \int_C \varphi(x, y, z) \, dx + \hat{\mathbf{y}} \int_C \varphi(x, y, z) \, dy + \hat{\mathbf{z}} \int_C \varphi(x, y, z) \, dz.$$

So that is true in the Cartesian systems, where unit vectors are constant in both magnitude and direction.

# *Example 1.10.1 PATH-DEPENDENT WORK*

The force exerted on a body is $\mathbf{F} = -\hat{\mathbf{x}}y + \hat{\mathbf{y}}x$. The problem is to calculate the work done going from the origin to the point (1, 1):




$$W = \int_{0,0}^{1,1} \mathbf{F} \cdot d\mathbf{r} = \int_{0,0}^{1,1} (-y\,dx + x\,dy).$$

Separating the two integrals, we obtain

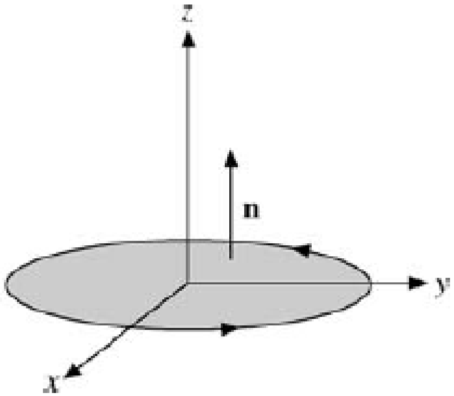
$$W = -\int_0^1 y\,dx + \int_0^1 x\,dy. \tag{1.95c}$$

The first integral cannot be evaluated until we specify the values of $y$ as $x$ ranges from 0 to 1. Likewise, the second integral requires $x$ as a function of $y$. Consider first the path shown in Fig. 1.25. Then

$$W = -\int_0^1 0\,dx + \int_0^1 1\,dy = 1, \tag{1.95d}$$

since $y = 0$ along the first segment of the path and $x = 1$ along the second. If we select the path $[x = 0, 0 \leqslant y \leqslant 1]$ and $[0 \leqslant x \leqslant 1, y = 1]$, then Eq. (1.95c) gives $W = -1$. For this force the work done depends on the choice of path.

# Surface Integrals




•Surface integrals may appear inthe forms

$$\int \varphi\, d\sigma, \qquad \int \mathbf{V}\cdot d\sigma, \qquad \int \mathbf{V}\times d\sigma.$$

•The dot product is by far the most commonly encountered form. The surface integral ∫V · dσ may be interpreted as a flow or flux through the given surface

•The element of area also being a vector, dσ . Often this area element is written dσ =ndA, in which n is a unit (normal) vector to indicate the positive direction

•If the surface is a closed surface, we agree to take the outward normal as positive

•If the surface is an open surface, the positive normal depends on the direction in which the perimeter of the open surface is traversed

# Volume Integrals

Volume integrals are somewhat simpler, for the volume element $d\tau$ *is a scalar quantity*

$$\int_V \mathbf{V}\, d\tau = \hat{\mathbf{x}} \int_V V_x\, d\tau + \hat{\mathbf{y}} \int_V V_y\, d\tau + \hat{\mathbf{z}} \int_V V_z\, d\tau,$$

reducing the vector integral to a vector sum of scalar integrals.

# Integral Definitions of Gradient, Divergence, and Curl

In these three equations ∫*dτ is the volume of a small region of space and d***σ is the vector** area element of this volume

$$\nabla \varphi = \lim_{\int d\tau \to 0} \frac{\int \varphi \, d\boldsymbol{\sigma}}{\int d\tau},$$

$$\nabla \cdot \mathbf{V} = \lim_{\int d\tau \to 0} \frac{\int \mathbf{V} \cdot d\boldsymbol{\sigma}}{\int d\tau},$$

$$\nabla \times \mathbf{V} = \lim_{\int d\tau \to 0} \frac{\int d\boldsymbol{\sigma} \times \mathbf{V}}{\int d\tau}.$$

# 1.11 GAUSS' THEOREM

$$\int_S \mathbf{V} \cdot d\sigma = \int_V \nabla \cdot \mathbf{V} \, d\tau .$$

the surface integral of a vector over a closed surface equals the volume integral of the divergence of that vector integrated over the volume enclosed by the surface.

## 1.12 STOKES' THEOREM

Here we consider a relation between the surface integral of a derivative of a function and the line integral of the function, the path of integration being the perimeter bounding the surface.

$$\oint \mathbf{V} \cdot d\boldsymbol{\lambda} = \int_S \nabla \times \mathbf{V} \cdot d\boldsymbol{\sigma} .$$

This is Stokes' theorem. The surface integral on the right is over the surface bounded

by the perimeter or contour, for the line integral on the left.

Actually we need only

demand that the curl of V(x, y, z) exist and that it be integrable over the surface

Stokes' theorem obviously applies to an open surface by the substitution V = a$\phi$, V = a×P in which a is a vector of constant magnitude and of constant direction,

We get Alternate Forms of Stokes' Theorem

$$\int_S d\boldsymbol{\sigma} \times \nabla\varphi = \oint_{\partial S} \varphi \, d\boldsymbol{\lambda}$$

$$\int_S (d\boldsymbol{\sigma} \times \nabla) \times \mathbf{P} = \oint_{\partial S} d\boldsymbol{\lambda} \times \mathbf{P}.$$

*Example 1.12.1*

*we can integrateMaxwell's equation for* $\nabla \times E$*, to yield Faraday's induction law. Imagine moving a closed loop (∂S) of wire (of area S) across a magnetic induction field B. We integrate Maxwell's equation and use Stokes' theorem, yielding*

$$\int_{\partial S} \mathbf{E} \cdot d\mathbf{r} = \int_S (\nabla \times \mathbf{E}) \cdot d\sigma = -\frac{d}{dt} \int_S \mathbf{B} \cdot d\sigma = -\frac{d\Phi}{dt},$$

*which is Faraday's law. The line integral on the left-hand side represents the voltage induced in the wire loop, while the right-hand side is the change with time of the magnetic flux through the moving surface S of the wire.*

# 1.13 POTENTIAL THEORY

## Scalar Potential

The force F appearing as the negative gradient of a single-valued scalar potential is labeled a conservative force and ϕ a scalar potential that describes the force

$$\mathbf{F} = -\nabla\varphi, \qquad \nabla \times \mathbf{F} = 0 \qquad \oint \mathbf{F} \cdot d\mathbf{r} = 0,$$

Each of these three equations implies the other two

A single-valued scalar potential function ϕ exists if and only if F is irrotational or the work done around every closed loop is zero.

## Vector Potential

*When a vector* ***B is solenoidal ;*** **$\nabla \cdot B = 0$,** ***a*** *vector potential* ***A exists such that***

$$\boldsymbol{B} = \nabla \times \boldsymbol{A}$$

## *Exercises*

***1.9.3***

***1.9.7***

***1.9.12***

***1.10.2***

***1.11.2***

***1.12.1***

Deadline 10- 2- 1442