1.5 TRIPLE SCALAR PRODUCT, TRIPLE VECTOR PRODUCT

Triple Scalar Product

$$\mathbf{A} \cdot \mathbf{B} \times \mathbf{C} = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}.$$

There is a high degree of symmetry in the component expansion

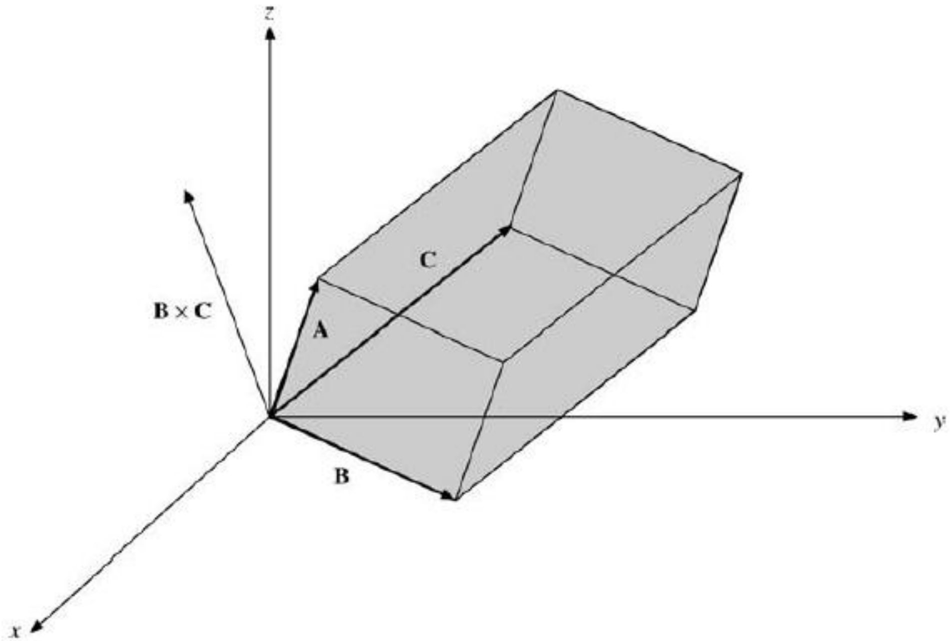
So that the determinant changes sign if any two rows are interchanged

$$\mathbf{A} \cdot \mathbf{B}\times\mathbf{C} = \mathbf{B} \cdot \mathbf{C}\times\mathbf{A} = \mathbf{C} \cdot \mathbf{A}\times\mathbf{B}$$

$$= -\mathbf{A} \cdot \mathbf{C}\times\mathbf{B} = -\mathbf{C} \cdot \mathbf{B}\times\mathbf{A} = -\mathbf{B} \cdot \mathbf{A}\times\mathbf{C}$$

Further, the dot and the cross may be interchanged,

$$\mathbf{A} \cdot \mathbf{B}\times\mathbf{C} = \mathbf{A}\times\mathbf{B} \cdot \mathbf{C}$$



|B×C| = ***BC sin ϑ***

= area of parallelogram base.

A · B×C = volume of parallelepiped defined by A*,B, and C.*

Triple Vector Product

The second triple product of interest is **A×*(B×C), which is a vector***

Example 1.5.1 A TRIPLE VECTOR PRODUCT

For the vectors

$$\mathbf{A} = \hat{\mathbf{x}} + 2\hat{\mathbf{y}} - \hat{\mathbf{z}} = (1, 2, -1), \quad \mathbf{B} = \hat{\mathbf{y}} + \hat{\mathbf{z}} = (0, 1, 1), \quad \mathbf{C} = \hat{\mathbf{x}} - \hat{\mathbf{y}} = (0, 1, 1),$$

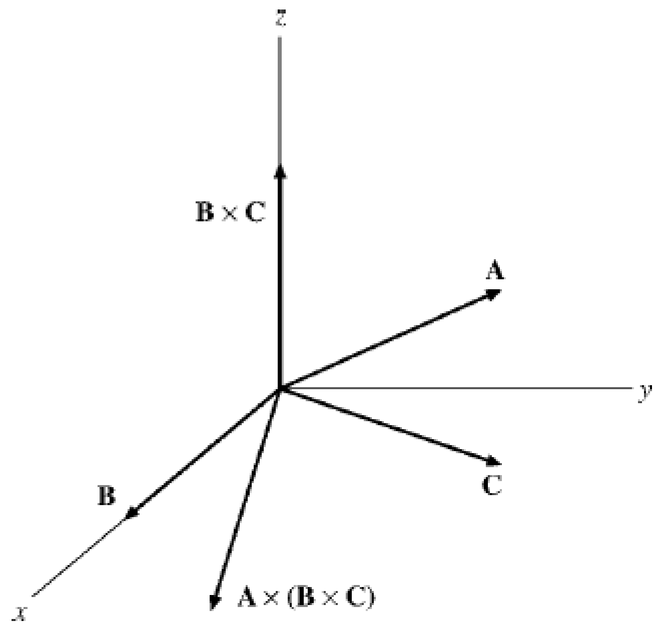
$$\mathbf{B} \times \mathbf{C} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ 0 & 1 & 1 \\ 1 & -1 & 0 \end{vmatrix} = \hat{\mathbf{x}} + \hat{\mathbf{y}} - \hat{\mathbf{z}},$$

and

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ 1 & 2 & -1 \\ 1 & 1 & -1 \end{vmatrix} = -\hat{\mathbf{x}} - \hat{\mathbf{z}} = -(\hat{\mathbf{y}} + \hat{\mathbf{z}}) - (\hat{\mathbf{x}} - \hat{\mathbf{y}})$$

$$= -\mathbf{B} - \mathbf{C}.$$

taking a geometric approach



$$\mathbf{A}\times(\boldsymbol{B}\times\boldsymbol{C}) = u\boldsymbol{B} + v\boldsymbol{C}.$$

$$\boxed{\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = [\mathbf{B}(\mathbf{A}\cdot\mathbf{C}) - \mathbf{C}(\mathbf{A}\cdot\mathbf{B})];}$$

known as the **BAC–CAB rule**

Homework 1	***Exercises 1.1.1 to 1.1.7*** ***And 1.2.1, 1.2.2***
Homework 2	***Exercises 1.3.3, 1.3.4, 1.3.6*** ***And 1.4.4, 1.4.5, 1.4.7***
Homework 3	***Exercises*** **1.5.1 , 1.5.10, 1.5.12**
Homework 4	
Homework 5	
Homework 6	
Homework 7	
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- **File Name : name- homework No**
- **Write your name inside**
- **One PDF file for each homework**
- **Submit before deadline**

VECTOR DIFFERENTIAL OPERATOR

1.6 GRADIENT, ∇

***(∇ϕ) is gradient of the scalar ϕ, whereas (∇) itself is a vector differential* operator** (*ϕ is* differentiable function of position*).*

$$\nabla \varphi = \hat{\mathbf{x}} \frac{\partial \varphi}{\partial x} + \hat{\mathbf{y}} \frac{\partial \varphi}{\partial y} + \hat{\mathbf{z}} \frac{\partial \varphi}{\partial z}$$

$$\nabla = \hat{\mathbf{x}} \frac{\partial}{\partial x} + \hat{\mathbf{y}} \frac{\partial}{\partial y} + \hat{\mathbf{z}} \frac{\partial}{\partial z}$$

- The relation between a force field and a potential field, force which holds for both gravitational and electrostatic fields

F=−∇*(potential V),*

- Because the total variation *is the work done against the force along the path dr,*

$$dV = \nabla V \cdot \mathbf{dr} = -\mathbf{F} \cdot \mathbf{dr}$$

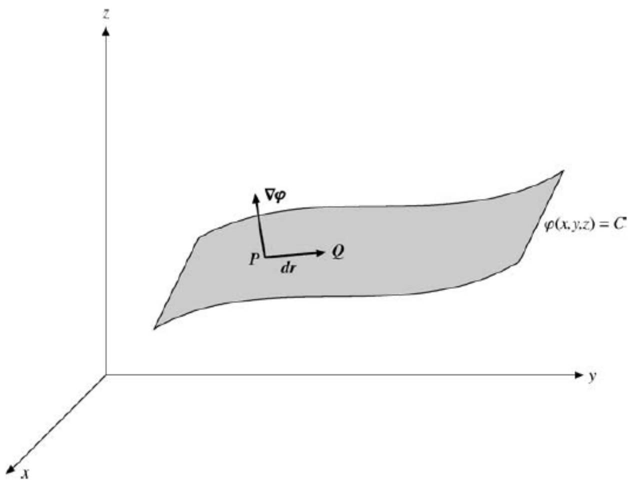
we recognize the physical meaning of the potential (difference) as work and energy.

A Geometrical Interpretation

$$\nabla\varphi \cdot d\mathbf{r} = \frac{\partial\varphi}{\partial x}dx + \frac{\partial\varphi}{\partial y}dy + \frac{\partial\varphi}{\partial z}dz = d\varphi,$$

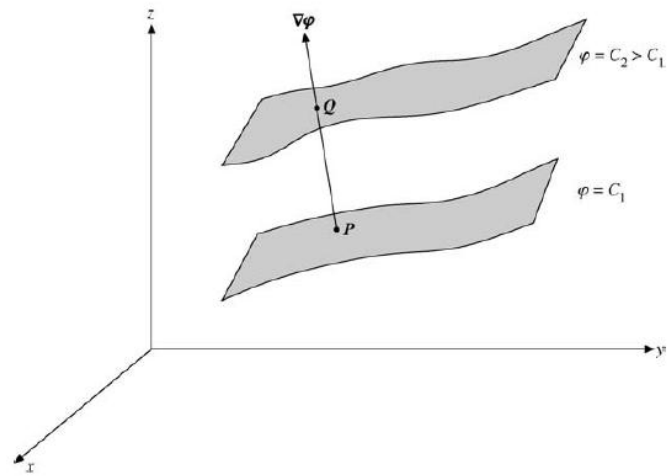
Let (dr) take us from P and Q to be two points on a surface $\phi(x, y, z) = C$, a constant

$d\phi = (\nabla\phi) \cdot dr = 0$ (perpendicular)



now permit (*dr) to take us from one surface $\phi = C1$ to an adjacent surface $\phi = C2$*

$d\phi = C1 - C2 = (\nabla\phi) \cdot dr.$ (parallel)



This identifies $\nabla\phi$ *as a vector having the direction of the maximum space rate* of change of ϕ,

Example 1.6.1 THE GRADIENT OF A POTENTIAL $V(r)$

Let us calculate the gradient of $V(r) = V(\sqrt{x^2+y^2+z^2})$, so

$$\nabla V(r) = \hat{\mathbf{x}}\frac{\partial V(r)}{\partial x} + \hat{\mathbf{y}}\frac{\partial V(r)}{\partial y} + \hat{\mathbf{z}}\frac{\partial V(r)}{\partial z}.$$

Now, $V(r)$ depends on x through the dependence of r on x. Therefore[14]

$$\frac{\partial V(r)}{\partial x} = \frac{dV(r)}{dr} \cdot \frac{\partial r}{\partial x}.$$

From r as a function of x, y, z,

$$\frac{\partial r}{\partial x} = \frac{\partial (x^2+y^2+z^2)^{1/2}}{\partial x} = \frac{x}{(x^2+y^2+z^2)^{1/2}} = \frac{x}{r}.$$

Therefore

$$\frac{\partial V(r)}{\partial x} = \frac{dV(r)}{dr} \cdot \frac{x}{r}.$$

Permuting coordinates ($x \to y, y \to z, z \to x$) to obtain the y and z derivatives, we get

$$\boxed{\begin{aligned} \nabla V(r) &= (\hat{\mathbf{x}}x + \hat{\mathbf{y}}y + \hat{\mathbf{z}}z)\frac{1}{r}\frac{dV}{dr} \\ &= \frac{\mathbf{r}}{r}\frac{dV}{dr} = \hat{\mathbf{r}}\frac{dV}{dr}. \end{aligned}}$$

The gradient of a function of *r is a vector in the (positive or negative) radial direction.*

1.7 DIVERGENCE, ∇

The **divergence of a vector A is defined as the operation**

$$\nabla \cdot \mathbf{r} = \left(\hat{\mathbf{e}}_x \frac{\partial}{\partial x} + \hat{\mathbf{e}}_y \frac{\partial}{\partial y} + \hat{\mathbf{e}}_z \frac{\partial}{\partial z}\right) \cdot (\hat{\mathbf{e}}_x x + \hat{\mathbf{e}}_y y + \hat{\mathbf{e}}_z z)$$

$$= \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z},$$

$$= \textit{Scalar Field}$$

<u>Properties :</u>

$$\operatorname{div}(a\mathbf{F} + b\mathbf{G}) = a \operatorname{div} \mathbf{F} + b \operatorname{div} \mathbf{G}$$

$$\nabla \cdot (\varphi \mathbf{F}) = (\nabla \varphi) \cdot \mathbf{F} + \varphi (\nabla \cdot \mathbf{F}).$$

$$\nabla \cdot (\mathbf{F} \times \mathbf{G}) = (\nabla \times \mathbf{F}) \cdot \mathbf{G} - \mathbf{F} \cdot (\nabla \times \mathbf{G}).$$

If we have the special case of the divergence of a vector vanishing,

$$\nabla \cdot \mathbf{B} = 0$$

the vector **B is said to be solenoidal**

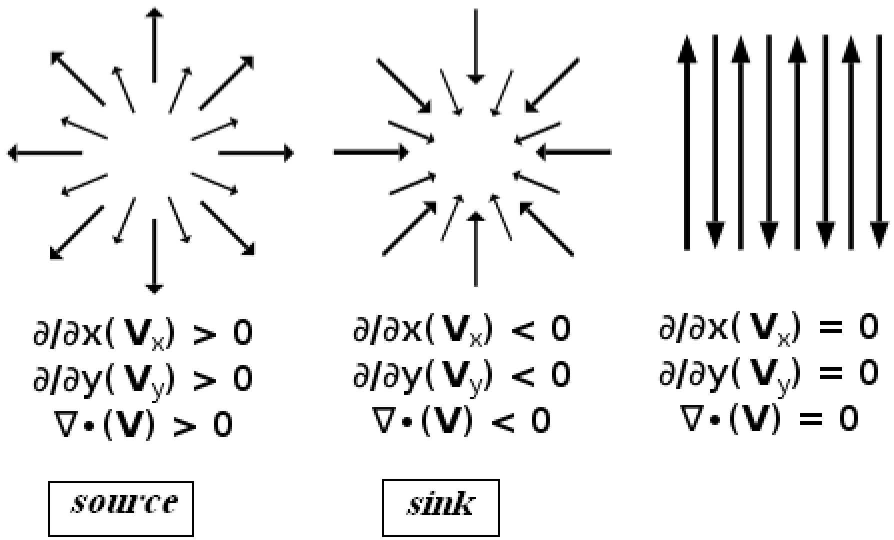
When a vector is solenoidal, it may be written as the curl of another vector known as the vector potential

A Physical Interpretation

A direct application is in the continuity equation for a compressible fluid

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0,$$

which states that a net flow out of the volume results in a decreased density inside the volume



The divergence of a vector field is often illustrated using the example of the velocity field of a fluid, a liquid or gas. A moving gas has a velocity that forms a vector field. If a gas is heated, it will expand in all directions. so there will be an outward flux of gas through the surface encloses the gas . So the velocity field will have positive divergence everywhere.

Similarly, if the gas is cooled, it will contract. There will be a net flow of gas volume inward through any closed surface. Therefore the velocity field has negative divergence everywhere.

In contrast in an unheated gas with a constant density, the volume rate of gas flowing into any closed surface must equal the volume rate flowing out, so the *net* flux of fluid through any closed surface is zero. Thus the gas velocity has zero divergence everywhere.

A field which has zero divergence everywhere is called solenoidal.

1.8 CURL, ∇×

- Another possible operation with the vector operator ∇ **is to cross it into a vector**

$$\nabla \times \mathbf{V} = \hat{\mathbf{x}}\left(\frac{\partial}{\partial y}V_z - \frac{\partial}{\partial z}V_y\right) + \hat{\mathbf{y}}\left(\frac{\partial}{\partial z}V_x - \frac{\partial}{\partial x}V_z\right) + \hat{\mathbf{z}}\left(\frac{\partial}{\partial x}V_y - \frac{\partial}{\partial y}V_x\right)$$

$$= \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_x & V_y & V_z \end{vmatrix},$$

- The determinant must be expanded **from the top down**
- If ∇ **is crossed into the product of a scalar and a vector,**

$$\nabla \times (f\mathbf{V}) = f\nabla \times \mathbf{V} + (\nabla f) \times \mathbf{V}$$

Example 1.8.2 CURL OF A CENTRAL FORCE FIELD

Calculate $\nabla \times (\mathbf{r} f(r))$.

by using $\nabla \times (f\mathbf{V}) = f\nabla \times \mathbf{V} + (\nabla f) \times \mathbf{V}$

$$\nabla \times \left(\mathbf{r} f(r)\right) = f(r)\nabla \times \mathbf{r} + \left[\nabla f(r)\right] \times \mathbf{r}.$$

First term is
$$\nabla \times \mathbf{r} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix} = 0.$$

Second, using $\nabla f(r) = \hat{\mathbf{r}}(df/dr)$, we obtain

$$\nabla \times \mathbf{r} f(r) = \frac{df}{dr}\hat{\mathbf{r}} \times \mathbf{r} = 0.$$

This vector product vanishes, since $\mathbf{r} = \hat{\mathbf{r}}r$ and $\hat{\mathbf{r}} \times \hat{\mathbf{r}} = 0$.

Whenever the curl of a vector V vanishes,

V is labeled **irrotational**. The most important physical examples of irrotational vectors are the gravitational and electrostatic forces

$$\nabla \times \mathbf{V} = 0,$$

•Using the gradient, divergence, and curl, and of course the *BAC–CAB rule, we may* construct or verify a large number of useful vector identities.

Remember that **∇ is a vector operator, a hybrid creature satisfying two sets of rules:**

1. vector rules, and
2. partial differentiation rules—including differentiation of a product.

For verification, complete expansion into Cartesian components is always a possibility . Sometimes if we use insight instead of routine shuffling of Cartesian components, the verification process can be shortened drastically.

Distributive properties

$$\nabla(\psi + \phi) = \nabla\psi + \nabla\phi$$
$$\nabla(\mathbf{A} + \mathbf{B}) = \nabla\mathbf{A} + \nabla\mathbf{B}$$
$$\nabla\cdot(\mathbf{A} + \mathbf{B}) = \nabla\cdot\mathbf{A} + \nabla\cdot\mathbf{B}$$
$$\nabla\times(\mathbf{A} + \mathbf{B}) = \nabla\times\mathbf{A} + \nabla\times\mathbf{B}$$

Product rule for multiplication by a scalar

$$\nabla(\psi\phi) = \phi\,\nabla\psi + \psi\,\nabla\phi$$
$$\nabla\cdot(\psi\mathbf{A}) = \psi\,\nabla\cdot\mathbf{A} + (\nabla\psi)\cdot\mathbf{A}$$
$$\nabla\times(\psi\mathbf{A}) = \psi\,\nabla\times\mathbf{A} + (\nabla\psi)\times\mathbf{A}$$
$$\nabla^2(fg) = f\,\nabla^2 g + 2\,\nabla f\cdot\nabla g + g\,\nabla^2 f$$

Quotient rule for division by a scalar

$$\nabla\left(\frac{\psi}{\phi}\right) = \frac{\phi\,\nabla\psi - \psi\,\nabla\phi}{\phi^2}$$

$$\nabla\cdot\left(\frac{\mathbf{A}}{\phi}\right) = \frac{\phi\,\nabla\cdot\mathbf{A} - \nabla\phi\cdot\mathbf{A}}{\phi^2}$$

$$\nabla\times\left(\frac{\mathbf{A}}{\phi}\right) = \frac{\phi\,\nabla\times\mathbf{A} - \nabla\phi\times\mathbf{A}}{\phi^2}$$

Divergence of curl is zero

$$\nabla\cdot(\nabla\times\mathbf{A}) = 0$$

Divergence of gradient is Laplacian

$$\nabla^2\psi = \nabla\cdot(\nabla\psi)$$

Divergence of divergence is undefined

$$\nabla\cdot(\nabla\cdot\mathbf{A}) = \text{undefined}$$

Curl of gradient is zero

$$\nabla\times(\nabla\phi) = \mathbf{0}$$

Curl of curl

$$\nabla\times(\nabla\times\mathbf{A}) = \nabla(\nabla\cdot\mathbf{A}) - \nabla^2\mathbf{A}$$

Curl of divergence is undefined

$$\nabla\times(\nabla\cdot\mathbf{A}) \text{ is undefined}$$