501فيز الفيزياء الرياضية 3(0+3)

توصيف المقرر

This course offers advanced knowledge in Vector Analysis, Vector Analysis in Curved Coordinates and Tensors, Functions of Complex variable I, Functions of Complex variable II, Differential equations, Sturm- Liouville Theory-Orthogonal Functions.

1. Topics to be Covered		
List of Topics	No. of Weeks	Contact hours
Vector Analysis	**3 weeks**	
Vector Analysis in Curved Coordinates and Tensors	**1 week**	
Functions of Complex variable I	**2 weeks**	
Functions of Complex variable II,	**3 weeks**	
Differential equations	**2 weeks**	
Sturm-Liouville Theory-	**2 weeks**	
Orthogonal Functions.	**1 week**	
Total	**14 weeks**	

Required Textbooks

Mathematical Methods for Physicists, G. Arfken 3rd Edition

Mathematical Methods for Physicists, Arfken, Weber, 6th Edition

Electronic Materials, Web Sites

https://ocw.mit.edu/courses/mathematics/18-152-introduction-to-partial-differential-equations-fall-2011/

https://ocw.mit.edu/resources/res-18-008-calculus-revisited-complex-variables-differential-equations and-linear-algebra-fall-2011/part-i/

https://ocw.mit.edu/resources/res-18-007-calculus-revisited-multivariable-calculus-fall-2011/part-ii/

VECTOR ANALYSIS

1.1 DEFINITIONS, ELEMENTARY APPROACH

scalar quantities : that have
magnitude only (mass, time, and temperature)

vector quantities : that have magnitude and direction (displacement, velocity, acceleration, force, momentum)

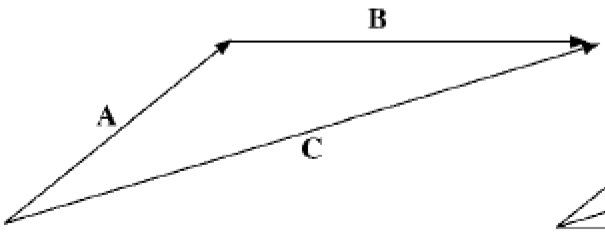
Our vector may be conveniently represented by an arrow, with length proportional to the magnitude. The direction of the arrow gives the direction of the vector

Vectors enter physics in two distinct forms.

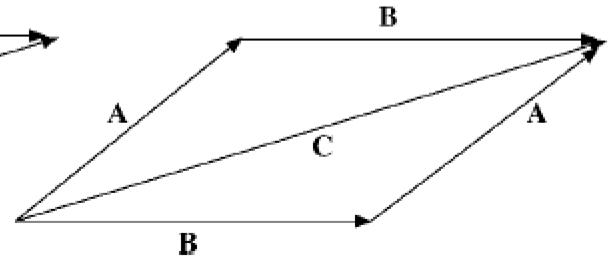
(1) Vector **A may represent a single force** acting at a single point

(2) Vector **A may be defined over some extended region** and being a function of position (**vector field)**

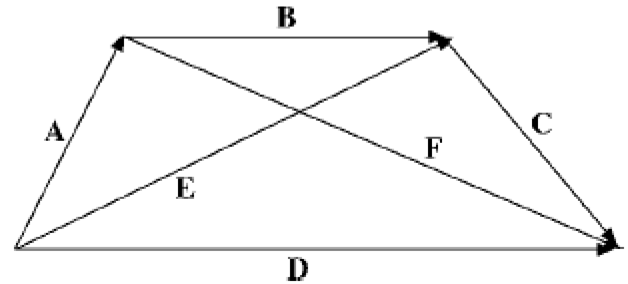
Geometrical representation



Triangle law of vector addition.



vector addition is **commutative**



Vector addition is **associative**

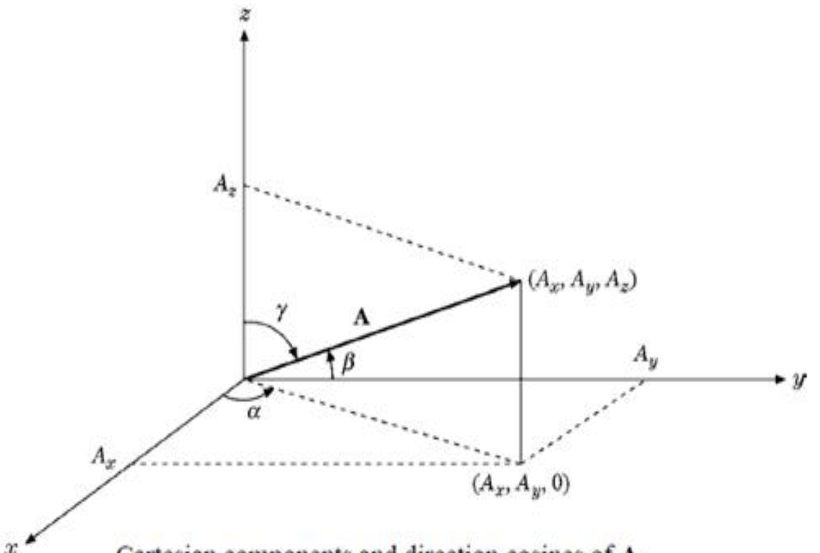
$$\mathbf{C} = \mathbf{A}+\mathbf{B} = \mathbf{B}+\mathbf{A},$$

$$(A + B)+C = A + (B+ C)$$

$$\mathbf{A} - \mathbf{B} = \mathbf{A} + (-B).$$

Note that the vectors are treated as geometrical objects that are independent of any coordinate system

Algebraic representation



Cartesian components and direction cosines of **A**.

$$|\mathbf{A}| = \left(A_x^2 + A_y^2 + A_z^2\right)^{1/2}$$

Ax,Ay , and Az are known as the **components of A or the projections** of **A,**
cosα, cosβ, and cos γ are called the **direction cosines**
with **$\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1$.**
A = ˆxAx + ˆyAy + ˆzAz.
ˆx = (1, 0, 0), ˆy = (0, 1, 0) and ˆz = (0, 0, 1)
A = 0, then Ax = Ay = Az = 0
A±B = ˆx(Ax ±Bx)+ ˆy(Ay ± By)+ ˆz(Az ±Bz).

1.2 ROTATION OF THE COORDINATE AXES

There is an important physical basis for our development of a new definition:

we describe our physical world by mathematics

but it and any physical predictions we may make

must be **independent for our** mathematical analysis.

We assume that space is **isotropic: that is, there is no** preferred direction or all directions equivalent.

Then the physical system being analyzed or the physical law being enunciated cannot and must no depend on our choice or **orientation of the coordinate axes.**

In this section two more definitions are presented.

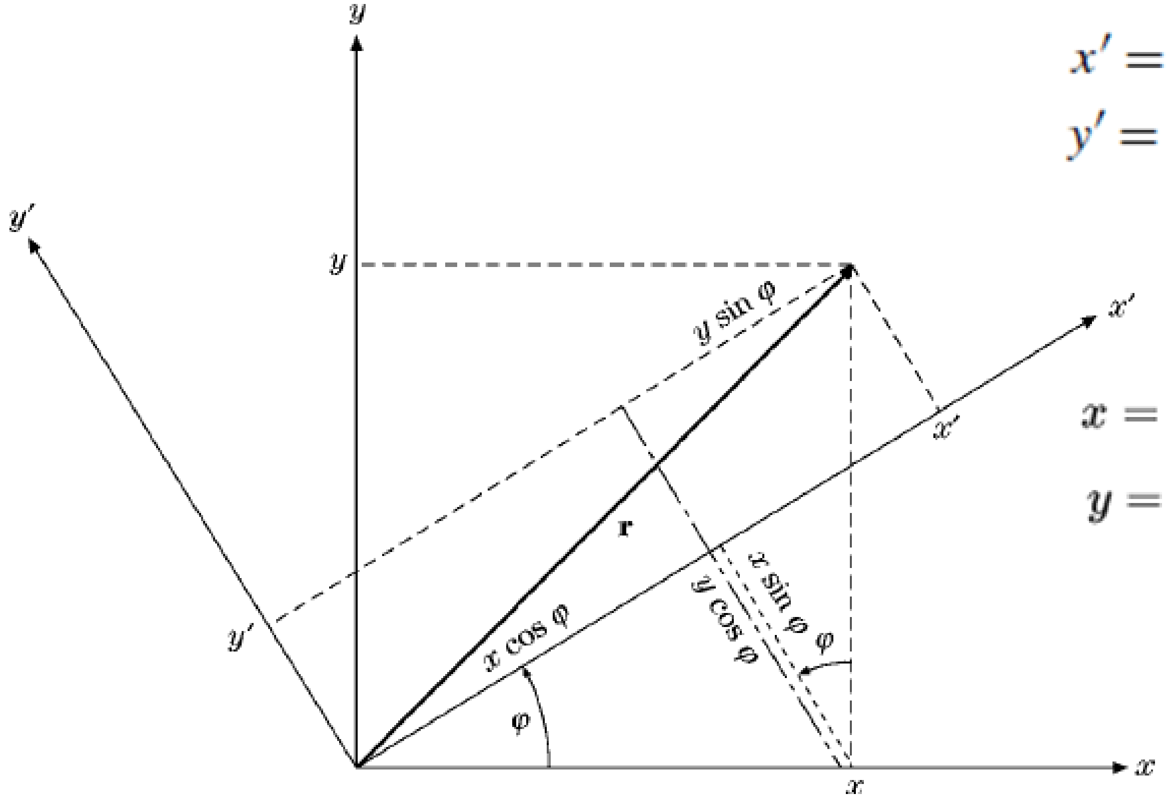
First, the vector field is defined

in terms of the behavior of its components under rotation of the coordinate axes. This

approach leads into the tensor analysis

Second, the component definition is generalized according to the mathematician's concepts of vector and vector space. This

approach leads to function spaces, including the Hilbert space.



$$x' = x\cos\varphi + y\sin\varphi,$$
$$y' = -x\sin\varphi + y\cos\varphi.$$

$$x = x'\cos\varphi - y'\sin\varphi$$
$$y = x'\sin\varphi + y'\cos\varphi$$

the x-, y-coordinates are rotated counterclockwise through an angle ϕ, **keeping r, fixed**

$$A'_x = A_x\cos\varphi + A_y\sin\varphi,$$
$$A'_y = -A_x\sin\varphi + A_y\cos\varphi,$$

the transformation relations, are a guarantee

that the entity **A is independent of the rotation of the coordinate system** more compact Notation

$$x \to x_1$$
$$y \to x_2$$

$$a_{11} = \cos\varphi, \qquad a_{12} = \sin\varphi,$$
$$a_{21} = -\sin\varphi, \qquad a_{22} = \cos\varphi.$$

$$x_1' = a_{11}x_1 + a_{12}x_2,$$
$$x_2' = a_{21}x_1 + a_{22}x_2.$$

$$x_i' = \sum_{j=1}^{2} a_{ij}x_j, \qquad i = 1, 2.$$

The generalization to three, four, or *N dimensions is now simple. The set of N quantities Vj is said to be the components of an N-dimensional vector* ***V if and only if their values*** relative to the rotated coordinate axes are given by

$$V_i' = \sum_{j=1}^{N} a_{ij}V_j, \qquad i = 1, 2, \ldots, N.$$

we may write (Cartesian coordinates)

$$a_{ij} = \frac{\partial x'_i}{\partial x_j}.$$

Using the inverse rotation ($\varphi \to -\varphi$) yields

$$x_j = \sum_{i=1}^{2} a_{ij} x'_i \qquad \text{or} \qquad \frac{\partial x_j}{\partial x'_i} = a_{ij}.$$

Note that these are **partial derivatives**. becomes

$$V'_i = \sum_{j=1}^{N} \frac{\partial x'_i}{\partial x_j} V_j = \sum_{j=1}^{N} \frac{\partial x_j}{\partial x'_i} V_j.$$

The direction cosines a_{ij} satisfy an **orthogonality condition**

$$\sum_i a_{ij} a_{ik} = \delta_{jk} \qquad \text{or, equivalently,} \qquad \sum_i a_{ji} a_{ki} = \delta_{jk}.$$

Here, the symbol δ_{jk} is the Kronecker delta, defined by

$$\begin{aligned} \delta_{jk} &= 1 \qquad \text{for} \qquad j = k, \\ \delta_{jk} &= 0 \qquad \text{for} \qquad j \neq k. \end{aligned}$$

in general form

$$\sum_i \frac{\partial x_j}{\partial x_i'} \frac{\partial x_k}{\partial x_i'} = \sum_i \frac{\partial x_j}{\partial x_i'} \frac{\partial x_i'}{\partial x_k} = \frac{\partial x_j}{\partial x_k}.$$

The last step follows by the standard rules for partial differentiation, assuming that x_j is a function of x_1', x_2', x_3', and so on.

The final result, $\partial x_j/\partial x_k$, is equal to δ_{jk}, since x_j and x_k as coordinate lines ($j \neq k$) are assumed to be perpendicular (two or three dimensions) or orthogonal (for any number of dimensions).

Equivalently, we may assume that x_j and x_k ($j \neq k$) are totally independent variables.

Vectors and Vector Space

It is customary in mathematics to label an ordered triple of real numbers (*x1, x2, x3) a* **vector x.** ***The collection of all*** such vectors **form** a three-dimensional real **vector space**

(if obeying the properties that follow)

1. Vector equality: $\mathbf{x} = \mathbf{y}$ means $x_i = y_i$, $i = 1, 2, 3$.
2. Vector addition: $\mathbf{x} + \mathbf{y} = \mathbf{z}$ means $x_i + y_i = z_i$, $i = 1, 2, 3$.
3. Scalar multiplication: $a\mathbf{x} \leftrightarrow (ax_1, ax_2, ax_3)$ (with a real).
4. Negative of a vector: $-\mathbf{x} = (-1)\mathbf{x} \leftrightarrow (-x_1, -x_2, -x_3)$.
5. Null vector: There exists a null vector $\mathbf{0} \leftrightarrow (0, 0, 0)$.

Since our vector components are real (or complex) numbers, the following properties also hold:

1. Addition of vectors is commutative: $\mathbf{x} + \mathbf{y} = \mathbf{y} + \mathbf{x}$.
2. Addition of vectors is associative: $(\mathbf{x} + \mathbf{y}) + \mathbf{z} = \mathbf{x} + (\mathbf{y} + \mathbf{z})$.
3. Scalar multiplication is distributive:

$$a(\mathbf{x} + \mathbf{y}) = a\mathbf{x} + a\mathbf{y}, \qquad \text{also} \qquad (a + b)\mathbf{x} = a\mathbf{x} + b\mathbf{x}.$$

4. Scalar multiplication is associative: $(ab)\mathbf{x} = a(b\mathbf{x})$.

the concept of vectors

presented here may be generalized to

(1) complex quantities,

(2) functions, and

(3) an infinite number of components.

This leads to infinite-dimensional function spaces, the Hilbert spaces, which are important in modern quantum theory.

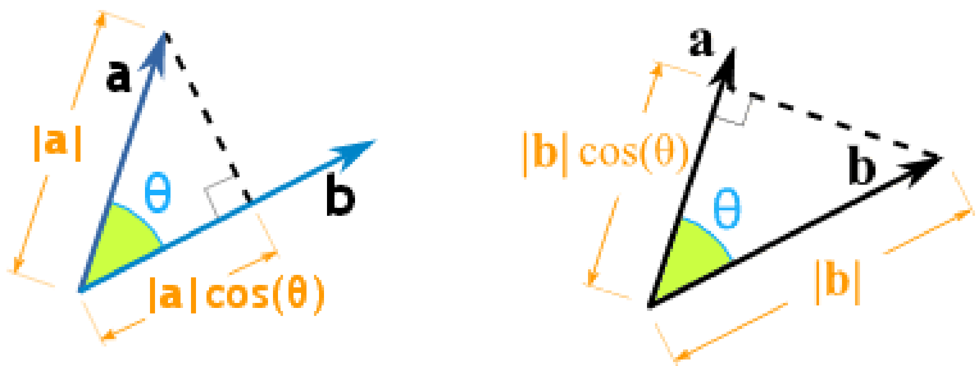
Exercises

1.1.1   to 1.1.7

And

1.2.1, 1.2.2

1.3 SCALAR OR DOT PRODUCT



$$\mathbf{A} \cdot \mathbf{B} \equiv A_B B = A B_A = AB\cos\theta.$$

$$\mathbf{A} \cdot \mathbf{B} \equiv \sum_i B_i A_i = \sum_i A_i B_i = \mathbf{B} \cdot \mathbf{A}.$$

$$Ax = A\cos\alpha \equiv \mathbf{A} \cdot \hat{x},$$

$$Ay = A\cos\beta \equiv \mathbf{A} \cdot \hat{y},$$

$$Az = A\cos\gamma \equiv \mathbf{A} \cdot \hat{z}.$$

General properties the scalar product

- obey the distributive and associative laws

$$\mathbf{A} \cdot (B+C) = A \cdot B + A \cdot C$$

$$\mathbf{A} \cdot (yB) = (yA) \cdot B = yA \cdot B,$$

- ***orthonormality***

$$\hat{x} \cdot \hat{x} = \hat{y} \cdot \hat{y} = \hat{z} \cdot \hat{z} = 1,$$

$$x \cdot \hat{y} = \hat{x} \cdot \hat{z} = \hat{y} \cdot \hat{z} = 0.$$

$$\mathbf{e}_m \cdot \mathbf{e}_n = \delta_{mn}. \qquad m = 1, 2, 3,$$

For *m = n the unit vectors* $\mathbf{e}_m$ ***and en are orthogonal. For m = n each vector is normalized*** to unity, that is, has unit magnitude.

The set $\mathbf{e}_m$ ***is said to be <u>orthonormal</u>***

•the scalar product is indeed a scalar quantity

It remains invariant under the rotation of the coordinate system.

$$\sum_k A'_k B'_k = \sum_i A_i B_i ,$$

•In a similar approach that exploits this concept of invariance

$$\text{take } \mathbf{C} = \mathbf{A} + \mathbf{B}$$

$$\mathbf{C} \cdot \mathbf{C} = (\mathbf{A} + \mathbf{B}) \cdot (\mathbf{A} + \mathbf{B}) = \mathbf{A} \cdot \mathbf{A} + \mathbf{B} \cdot \mathbf{B} + 2\mathbf{A} \cdot \mathbf{B} = C^2$$

$$\mathbf{A} \cdot \mathbf{B} = \frac{1}{2}(C^2 - A^2 - B^2), \qquad \text{invariant.}$$

Which is really another form of the law of cosines

$$C^2 = A^2 + B^2 + 2AB\cos\theta .$$

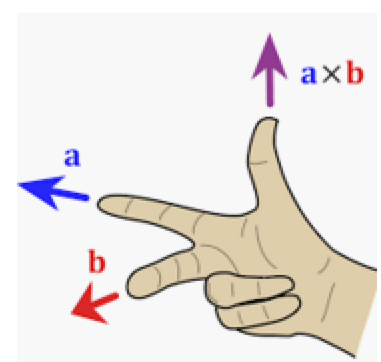
1.4 VECTOR OR CROSS PRODUCT

•For convenience in treating problems relating to quantities such as angular momentum, torque, and angular velocity,

•we define the vector product, or cross product

$$\mathbf{C} = \mathbf{A} \times \mathbf{B}, \qquad \text{with } C = AB\sin\theta.$$

C is now a vector, and we assign it a direction perpendicular to the plane of **A and B**

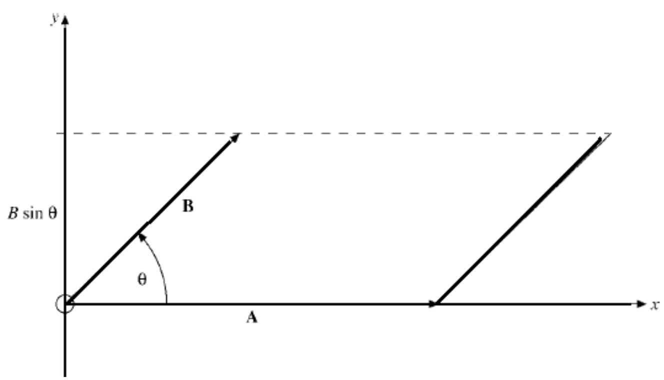


•Using the decomposition of **A and B into their Cartesian components**

$$\begin{aligned}\mathbf{A}\times\mathbf{B}\equiv\mathbf{C}=(C_x,C_y,C_z)&=(A_x\hat{\mathbf{x}}+A_y\hat{\mathbf{y}}+A_z\hat{\mathbf{z}})\times(B_x\hat{\mathbf{x}}+B_y\hat{\mathbf{y}}+B_z\hat{\mathbf{z}})\\&=(A_xB_y-A_yB_x)\hat{\mathbf{x}}\times\hat{\mathbf{y}}+(A_xB_z-A_zB_x)\hat{\mathbf{x}}\times\hat{\mathbf{z}}\\&\quad+(A_yB_z-A_zB_y)\hat{\mathbf{y}}\times\hat{\mathbf{z}}\end{aligned}$$

$$\mathbf{C}=\begin{vmatrix}\hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}}\\ A_x & A_y & A_z\\ B_x & B_y & B_z\end{vmatrix}\equiv\hat{\mathbf{x}}\begin{vmatrix}A_y & A_z\\ B_y & B_z\end{vmatrix}-\hat{\mathbf{y}}\begin{vmatrix}A_x & A_z\\ B_x & B_z\end{vmatrix}+\hat{\mathbf{z}}\begin{vmatrix}A_x & A_y\\ B_x & B_y\end{vmatrix}$$

•As a vector, **A×B is the area of the parallelogram defined by A and B, with** the area vector normal to the plane of the parallelogram



General properties the vector product

- ***anticommutation***

$$\mathbf{A}\times\mathbf{B}=-\mathbf{B}\times\mathbf{A},$$

- from this property of cross product we have

$$\hat{\mathbf{x}}\times\hat{\mathbf{x}}=\hat{\mathbf{y}}\times\hat{\mathbf{y}}=\hat{\mathbf{z}}\times\hat{\mathbf{z}}=0,$$

$$\begin{aligned}
&\hat{\mathbf{x}}\times\hat{\mathbf{y}}=\hat{\mathbf{z}}, &&\hat{\mathbf{y}}\times\hat{\mathbf{z}}=\hat{\mathbf{x}}, &&\hat{\mathbf{z}}\times\hat{\mathbf{x}}=\hat{\mathbf{y}},\\
&\hat{\mathbf{y}}\times\hat{\mathbf{x}}=-\hat{\mathbf{z}}, &&\hat{\mathbf{z}}\times\hat{\mathbf{y}}=-\hat{\mathbf{x}}, &&\hat{\mathbf{x}}\times\hat{\mathbf{z}}=-\hat{\mathbf{y}}.
\end{aligned}$$

And

$$\mathbf{A}\cdot(\mathbf{A}\times\mathbf{B})=\mathbf{B}\cdot(\mathbf{A}\times\mathbf{B})=0.$$

- the linearity of the cross product

$$\mathbf{A}\times(\boldsymbol{B}+\boldsymbol{C})=\boldsymbol{A}\times\boldsymbol{B}+\boldsymbol{A}\times\boldsymbol{C},$$

$$(\boldsymbol{A}+\boldsymbol{B})\times\boldsymbol{C}=\boldsymbol{A}\times\boldsymbol{C}+\boldsymbol{B}\times\boldsymbol{C},$$

$$\mathbf{A}\times(y\boldsymbol{B})=y\boldsymbol{A}\times\boldsymbol{B}=(y\boldsymbol{A})\times\boldsymbol{B}$$

•Combine vector and scalar product

$$
\begin{aligned}
(\mathbf{A} \times \mathbf{B}) \cdot (\mathbf{A} \times \mathbf{B}) &= A^2 B^2 - (\mathbf{A} \cdot \mathbf{B})^2 \\
&= A^2 B^2 - A^2 B^2 \cos^2 \theta \\
&= A^2 B^2 \sin^2 \theta .
\end{aligned}
$$

The first step may be verified by expanding out in component form

Where as the last step may be verified by the definition of vector product

$$C = AB \sin \vartheta$$

•There still remains the problem of verifying that **C = A × B is indeed a vector, that** is, that it obeys the vector transformation law

$$C'_i = A'_j B'_k - A'_k B'_j = \sum_l a_{jl} A_l \sum_m a_{km} B_m - \sum_l a_{kl} A_l \sum_m a_{jm} B_m$$

$$C'_3 = a_{33} A_1 B_2 + a_{32} A_3 B_1 + a_{31} A_2 B_3 - a_{33} A_2 B_1 - a_{32} A_1 B_3 - a_{31} A_3 B_2$$

$$= a_{31} C_1 + a_{32} C_2 + a_{33} C_3$$

$$= \sum_n a_{3n} C_n.$$

•It should be mentioned here that this **vector nature of the cross product** is an accident associated with the **three-dimensional nature of ordinary space**

But

•What about division by a vector?

• It turns out that the ratio **B*/A is not uniquely specified*** unless **A and B are also required to be parallel.**

•**Hence division of one vector by another <u>is not defined</u>.**

Exercises

1.3.3, 1.3.4, 1.3.6

And

1.4.4, 1.4.5, 1.4.7