5.1 + 5.2

1-

Let $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$. Show that A is diagonalizable and find the matrix P that diagonalizes A.

Solution:

Answer: Observe that:

$$0 = \det(\lambda I - A) = \begin{vmatrix} \lambda - 1 & -2 \\ -2 & \lambda - 1 \end{vmatrix}$$
$$= (\lambda - 1)^2 - 4 = \lambda^2 - 2\lambda + 1 - 4$$
$$= \lambda^2 - 2\lambda - 3 = (\lambda - 3)(\lambda + 1)$$

So, $\lambda = -1$ and $\lambda = 3$ are the eigenvalues of A and since they are different, A is diagonalizable. Now, we will find the eigenvectors by the equation $(\lambda I - A)x = 0$. When $\lambda = -1$, observe that

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$$\begin{bmatrix} \lambda - 1 & -2 \\ -2 & \lambda - 1 \end{bmatrix} = \begin{bmatrix} -2 & -2 \\ -2 & -2 \end{bmatrix} \xrightarrow{(-1)R_{12}} \begin{bmatrix} -2 & -2 \\ 0 & 0 \end{bmatrix} \xrightarrow{\left(-\frac{1}{2}\right)R_1} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

So, $x = -y = -t$, where $t \in \mathbb{R}$ and $(x, y) = (-t, t) = t(-1,1)$. Hence, $(-1,1)$ is an eigenvector of A corresponding to $\lambda = -1$. When $\lambda = 3$, observe that

$$\begin{bmatrix} \lambda - 1 & -2 \\ -2 & \lambda - 1 \end{bmatrix} = \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} \xrightarrow{1R_{12}} \begin{bmatrix} 2 & -2 \\ 0 & 0 \end{bmatrix} \xrightarrow{\left(\frac{1}{2}\right)R_1} \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$$

So, $x = y = t$, where $t \in \mathbb{R}$ and $(x, y) = (t, t) = t(1,1)$. Hence, $(1,1)$ is an eigenvector of A corresponding to $\lambda = 3$. Therefore,

$$P = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$$

2-

x) If $A = \begin{pmatrix} 2 & 0 \\ 3 & -1 \end{pmatrix}$, then the eigenvalues of A^4 are:

a) 2,16 b) -1,8 c) 1,16 d) 4,16.

Solution:

C)

Note that diagonalizability is not a requirement in Theorem 5.2.3.

THEOREM 5.2.3 *If k is a positive integer, λ is an eigenvalue of a matrix A, and $\mathbf{x}$ is a corresponding eigenvector, then λ^k is an eigenvalue of A^k and $\mathbf{x}$ is a corresponding eigenvector.*

3-

Show that the matrix $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 2 & 2 \\ 3 & 2 & 1 \end{pmatrix}$ is diagonalizable and find an invertible matrix P such that $P^{-1}AP$ is a diagonal matrix.

Solution:

$$\begin{aligned} q_A(\lambda) &= \begin{vmatrix} 1-\lambda & 2 & 3 \\ 2 & 2-\lambda & 2 \\ 3 & 2 & 1-\lambda \end{vmatrix} = (2+\lambda)\begin{vmatrix} -1 & 2 & 3 \\ 0 & 2-\lambda & 2 \\ 1 & 2 & 1-\lambda \end{vmatrix} \\ &= (2+\lambda)\begin{vmatrix} -1 & 2 & 3 \\ 0 & 2-\lambda & 2 \\ 0 & 4 & 4-\lambda \end{vmatrix} = -(2+\lambda)\begin{vmatrix} -\lambda & 2 \\ \lambda & 4-\lambda \end{vmatrix} \\ &= -\lambda(2+\lambda)(\lambda-6). \end{aligned}$$

For $\lambda = 0$, $X_1 = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$. For $\lambda = -2$, $X_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$. For $\lambda = 6$, $X_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

$P = \begin{pmatrix} 1 & 1 & 1 \\ -2 & 0 & 1 \\ 1 & -1 & 1 \end{pmatrix}$, $D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 6 \end{pmatrix}$.