12 $f(z)$ has a pole of order m at z_0

we prove $\frac{f'(z)}{f(z)} = g(z)$ has a simple pole.

since f has a pole of order m at z_0

$\Rightarrow f(z) = \frac{h(z)}{(z-z_0)^m}$, h is analytic at z_0 and $h(z_0) \neq 0$

$$\Rightarrow f'(z) = \frac{h'(z)(z-z_0)^m - h(z)\, m (z-z_0)^{m-1}}{(z-z_0)^{2m}}$$

$$= \frac{h'(z)(z-z_0) - h(z)\, m}{(z-z_0)^{m+1}}$$

$$\Rightarrow g(z) = \frac{f'(z)}{f(z)} = \frac{h'(z)(z-z_0) - h(z)\, m}{(z-z_0)^{\cancel{m+1}}} \cdot \frac{\cancel{(z-z_0)^m}}{h(z)}$$

$$= \frac{h'(z)(z-z_0) - h(z)\, m}{h(z)\,(z-z_0)}$$

$= \frac{k(z)}{z-z_0}$, where $k(z) = \frac{h'(z)(z-z_0) - h(z)\, m}{h(z)}$

k is analytic at z_0 and

$k(z_0) = \frac{h'(z_0) \cdot 0 - h(z_0)\, m}{h(z_0)} \neq 0$

$= -m \neq 0$

$\Rightarrow g(z)$ is pole of order

m

The coeffecient of $(z-z_0)^{-1}$ is

$\boxed{k(z_0) = a_{-1} = -m}$