3

5 (a) $f(z) = \sum_{k=0}^{\infty} (k^3/3^k)\, z^k$

$f^{(6)}(0)$

$$f' = \sum_{k=0}^{\infty} (k^3/3^k)\, k z^{k-1} = \sum (k^4)/3^k\, z^{k-1}$$

$$f'' = \sum_{k=0}^{\infty} \frac{k^4 (k-1)}{3^k} z^{k-2}$$

$\vdots$

$$f^{(4)} = \sum_{k=0}^{\infty} \frac{k^4 (k-1)(k-2)(k-3)}{3^k} z^{k-4}$$

$\vdots$

$$f^{(6)} = \sum_{k=0}^{\infty} \frac{k^4 (k-1)(k-2)(k-3)(k-4)(k-5)}{3^k} z^{k-6}$$

$\cancel{f^{(6)}(0)} = 0+0+0+0+0+0+ \ \uparrow \ + \boxed{\ }\,z + \boxed{\ }\,z^2 + \ldots$

at $k=6$

$$f^{(6)}(0) = \frac{6^4 (5)(4)(3)(2)(1)}{3^6}$$

(b) $$\int_{|z|=1} e^z f(z)\, dz = \int \sum_{k=0}^{\infty} (k^3/3^k)\, e^z z^k \, dz$$

$$= \sum_{k=0}^{\infty} (k^3/3^k) \int_{|z|=1} e^z z^k \, dz$$

analytic on $|z|=1$

$= 0$

6 $f(z) = \begin{cases} \sin z/z & z \neq 0 \\ 1 & z = 0 \end{cases}$

$$\sin z = \sum_{j=0}^{\infty} \frac{(-1)^j z^{2j+1}}{(2j+1)!}$$

$$\Rightarrow \sin z/z = \sum_{j=0}^{\infty} \frac{(-1)^j z^{2j}}{(2j+1)!} = 1 - \frac{z}{3!} + \frac{z^4}{5!} - \ldots$$

$\because f(0) = 1 \implies f(z) = 1 - z/3! + \frac{z^4}{5!} - \ldots \quad \forall z$