

4 Vector Spaces

4.1 Real Vector Spaces

4.2 Subspaces

4.1 Real Vector Spaces

Definition

Vector Space

Let V be a set on which two operations (addition and scalar multiplication) are defined. **If the following ten axioms are satisfied** for every elements \mathbf{u} , \mathbf{v} , and \mathbf{w} in V and every scalars (real numbers) c and d , then **V is called a vector space**, and the **elements** in V are called **vectors**

Addition:

- (1) $\mathbf{u} + \mathbf{v}$ is in V
- (2) $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
- (3) $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$
- (4) V has a zero vector $\mathbf{0}$ such that for every \mathbf{u} in V , $\mathbf{u} + \mathbf{0} = \mathbf{u}$
- (5) For every \mathbf{u} in V , there is a vector in V denoted by $-\mathbf{u}$ such that $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$

Scalar multiplication:

- (6) $c\mathbf{u}$ is in V
- (7) $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$
- (8) $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$
- (9) $c(d\mathbf{u}) = (cd)\mathbf{u}$
- (10) $1(\mathbf{u}) = \mathbf{u}$

Notes

A vector space consists of four entities

a set of vectors, a set of real-number scalars, and two operations

V : nonempty set of vectors

c : any scalar

$+(\mathbf{u}, \mathbf{v}) = \mathbf{u} + \mathbf{v}$: vector addition

$\cdot(c, \mathbf{u}) = c\mathbf{u}$: scalar multiplication

$(V, +, \cdot)$ is called a vector space

The set V together with the definitions of vector addition and scalar multiplication satisfying the above ten axioms is called a **vector space**

Example

$M_{m \times n}$ is a Vector Space

Proof Recall that the usual matrix addition and scalar multiplication have properties for any $A, B, C \in M_{m \times n}$ and any $s, t \in \mathbb{R}$:

1. $A + B \in M_{m \times n}$, **closed** under addition.
2. $A + B = B + A$, addition is **commutative**
3. $(A + B) + C = A + (B + C)$, addition is **associative**
4. There exists a zero matrix O_{mn} , such that $A + O_{mn} = A$, additive **identity**.
5. There exists a matrix $-A \in M_{m \times n}$ such that $A + (-A) = O_{mn}$, additive **inverse**.
6. $sA \in M_{m \times n}$, **closed** under scalar multiplication.
7. $s(tA) = (st)A$, scalar multiplication is **associative**.
8. $(s + t)A = sA + tA$ **matrix distribution**.
9. $s(A + B) = sA + sB$, **scalar distribution**.
10. $1A = A$, **scalar multiplicative identity**.

An ordered n -tuple: a sequence of n real numbers (x_1, x_2, \dots, x_n)

R^n -space : the set of all ordered n -tuples

$n = 1$ R^1 -space = set of all real numbers
(R^1 -space can be represented geometrically by the x -axis)

$n = 2$ R^2 -space = set of all ordered pair of real numbers (x_1, x_2)
(R^2 -space can be represented geometrically by the xy -plane)

$n = 3$ R^3 -space = set of all ordered triple of real numbers (x_1, x_2, x_3)
(R^3 -space can be represented geometrically by the xyz -space)

For $\mathbf{u} = (u_1, u_2, \dots, u_n)$, $\mathbf{v} = (v_1, v_2, \dots, v_n)$ (two vectors in R^n)

Equality: $\mathbf{u} = \mathbf{v}$ if and only if $u_1 = v_1, u_2 = v_2, \dots, u_n = v_n$

Vector addition (the sum of \mathbf{u} and \mathbf{v}): $\mathbf{u} + \mathbf{v} = (u_1 + v_1, u_2 + v_2, \dots, u_n + v_n)$

Scalar multiplication (the scalar multiple of \mathbf{u} by c): $c\mathbf{u} = (cu_1, cu_2, \dots, cu_n)$

Note: This addition and scalar multiplication are called the **standard operations** for R^n .

Zero vector: $\mathbf{0} = (0, 0, \dots, 0)$

additive inverse: $-\mathbf{u} = (-u_1, -u_2, \dots, -u_n)$

Notes:

A vector $\mathbf{u} = (u_1, u_2, \dots, u_n)$ in R^n can be viewed as:

Use comma to separate components

a $1 \times n$ row matrix (row vector): $\mathbf{u} = [u_1 \ u_2 \ \cdots \ u_n]$

or

Use blank space to separate entries

a $n \times 1$ column matrix (column vector): $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$

※ Therefore, the operations of matrix addition and scalar multiplication generate the same results as the corresponding vector operations

Vector addition

$$\begin{aligned}\mathbf{u} + \mathbf{v} &= (u_1, u_2, \dots, u_n) + (v_1, v_2, \dots, v_n) \\ &= (u_1 + v_1, u_2 + v_2, \dots, u_n + v_n)\end{aligned}$$

Scalar multiplication

$$\begin{aligned}c\mathbf{u} &= c(u_1, u_2, \dots, u_n) \\ &= (cu_1, cu_2, \dots, cu_n)\end{aligned}$$

Regarded as $1 \times n$ row matrix

$$\begin{aligned}\mathbf{u} + \mathbf{v} &= [u_1 \ u_2 \ \dots \ u_n] + [v_1 \ v_2 \ \dots \ v_n] \\ &= [u_1 + v_1 \ u_2 + v_2 \ \dots \ u_n + v_n]\end{aligned} \qquad \begin{aligned}c\mathbf{u} &= c[u_1 \ u_2 \ \dots \ u_n] \\ &= [cu_1 \ cu_2 \ \dots \ cu_n]\end{aligned}$$

Regarded as $n \times 1$ column matrix

$$\begin{aligned}\mathbf{u} + \mathbf{v} &= \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \\ \vdots \\ u_n + v_n \end{bmatrix} \qquad c\mathbf{u} = c \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} = \begin{bmatrix} cu_1 \\ cu_2 \\ \vdots \\ cu_n \end{bmatrix}\end{aligned}$$

Example

R^n is a Vector Space

Proof By regarding R^n as row (or column) matrices i.e. $R^n = M_{1 \times n}$ (or $R^n = M_{n \times 1}$), the fact that R^n is vector space becomes a special case for that of matrices.

Example

The Zero Vector Space

Let V consist of a single object, which we denote by 0 , that is $V = \{0\}$. Define

$$0 + 0 = 0, \text{ and } c0 = 0 \text{ for any } c \in R.$$

It is easy to check all 10 axioms are satisfied. V is called the zero vector space.

Example

The Vector Space of infinite sequences R^∞

- The set of all infinite sequences $u = (u_1, u_2, u_3, \dots)$ is denoted by R^∞ .
- $u = (u_1, u_2, u_3, \dots)$, $v = (v_1, v_2, v_3, \dots)$ are said to be equal if $u_i = v_i, \forall i \geq 1$.
- For $u, v \in R^\infty$ and $c \in R$, addition and scalar multiplication are defined by:

$$u + v = (u_1, u_2, u_3, \dots) + (v_1, v_2, v_3, \dots) = (u_1 + v_1, u_2 + v_2, u_3 + v_3, \dots)$$

$$cu = (cu_1, cu_2, cu_3, \dots)$$

- This makes R^∞ into a vector space, where

$$0 = (0, 0, 0, \dots) \text{ and } -u = (-u_1, -u_2, -u_3, \dots)$$

Example

The Vector Space of real Valued functions $F(D)$

- Let $D \subseteq R$. The set of all real valued function $f: D \rightarrow R$ is denoted by $F(D)$.
- $f, g \in F(D)$ are said to be equal if $f(x) = g(x), \forall x \in D$.
- For $f, g \in F(D)$ and $c \in R$, addition and scalar multiplication are defined by:

$$(f + g)(x) = f(x) + g(x)$$

$$(cf)(x) = cf(x)$$

- This together with the properties of addition on R makes $F(D)$ into a vector space, where 0 and $-f$ are the functions:

$$0(x) = 0 \text{ and } (-f)(x) = -f(x)$$

Example**A Set That Is NOT a Vector Space**

If $V = \mathbb{Z}$ is the set of integers, with addition and scalar multiplication. Then V is not a vector space since

$$1 \in V, \text{ and } \frac{1}{2} \text{ is a real-number scalar}$$

$$\begin{array}{c} \left(\frac{1}{2}\right)(1) = \frac{1}{2} \notin V \quad (\text{it is not closed under scalar multiplication}) \\ \uparrow \quad \uparrow \quad \uparrow \\ \text{scalar} \quad \text{integer} \quad \text{noninteger} \end{array}$$

Example**A Set That Is NOT a Vector Space**

If $V = R^2$, with addition and scalar multiplication:

$$(u_1, u_2) + (v_1, v_2) = (u_1 + v_1, u_2 + v_2) \qquad c(u_1, u_2) = (cu_1, 0) \quad (\text{nonstandard definition})$$

The first nine axioms of the definition of a vector space are satisfied (check it), but NOT tenth axiom. Since for example, $1(1,1) = (1,0) \neq (1,1)$.

Theorem

Properties of additive identity and additive inverse

Let v be any element of a vector space V , and let c be any scalar. Then

$$(1) \quad 0\mathbf{v} = \mathbf{0}$$

$$(2) \quad c\mathbf{0} = \mathbf{0}$$

$$(3) \quad \text{If } c\mathbf{v} = \mathbf{0}, \text{ either } c = 0 \text{ or } \mathbf{v} = \mathbf{0}$$

$$(4) \quad (-1)\mathbf{v} = -\mathbf{v} \quad (\text{the additive inverse of } \mathbf{v} \text{ equals } ((-1)\mathbf{v}))$$

Proof.

$$(1) \quad 0\mathbf{v} = (0 + 0)\mathbf{v} \stackrel{(8)}{=} 0\mathbf{v} + 0\mathbf{v} \stackrel{\text{add}(-0\mathbf{v})}{\Rightarrow} 0\mathbf{v} + (-0\mathbf{v}) = (0\mathbf{v} + 0\mathbf{v}) + (-0\mathbf{v}) \stackrel{(5,3)}{\Rightarrow} 0 = 0\mathbf{v}$$

$$(2) \quad c\mathbf{0} \stackrel{(4)}{=} c(\mathbf{0} + \mathbf{0}) \stackrel{(7)}{=} c\mathbf{0} + c\mathbf{0}$$

$$\Rightarrow c\mathbf{0} + (-c\mathbf{0}) = (c\mathbf{0} + c\mathbf{0}) + (-c\mathbf{0}) \quad (\text{add } (-c\mathbf{0}) \text{ to both sides})$$

$$\stackrel{(3)}{\Rightarrow} c\mathbf{0} + (-c\mathbf{0}) = c\mathbf{0} + [c\mathbf{0} + (-c\mathbf{0})]$$

$$\stackrel{(5)}{\Rightarrow} \mathbf{0} = c\mathbf{0} + \mathbf{0} \quad \stackrel{(4)}{\Rightarrow} \mathbf{0} = c\mathbf{0}$$

(3) Prove by contradiction: Suppose that $c\mathbf{v} = \mathbf{0}$, but $c \neq 0$ and $\mathbf{v} \neq \mathbf{0}$

$$\mathbf{v} = \overset{(10)}{1}\mathbf{v} = \left(\frac{1}{c}c\right)\mathbf{v} = \overset{(9)}{\frac{1}{c}}(c\mathbf{v}) = \frac{1}{c}(\mathbf{0}) = \mathbf{0} \quad (\text{By the second property, } c\mathbf{0} = \mathbf{0})$$

$\Rightarrow \rightarrow \leftarrow \Rightarrow$ if $c\mathbf{v} = \mathbf{0}$, either $c = 0$ or $\mathbf{v} = \mathbf{0}$

$$(4) \quad 0\mathbf{v} = (1 + (-1))\overset{(8)}{\mathbf{v}} = 1\mathbf{v} + (-1)\mathbf{v}$$

$\Rightarrow \mathbf{0} = \mathbf{v} + (-1)\mathbf{v}$ (By the first property, $0\mathbf{v} = \mathbf{0}$)

$\overset{(5)}{\Rightarrow} (-1)\mathbf{v} = -\mathbf{v}$ (By comparing with Axiom (5), $(-1)\mathbf{v}$ is the additive inverse of \mathbf{v})

Exercise:

True or False?

A vector space must contain at least two vectors.

A) True.

B) False.

 Multiple Choice

Exercise:

True or False?

If \mathbf{u} is a vector and k is a scalar such that $k\mathbf{u} = \mathbf{0}$, then it must be true that $k = 0$.

A) True.

B) False.

 Multiple Choice

Exercise:

True or False?

The set of positive real numbers is a vector space if vector addition and scalar multiplication are the usual operations of addition and multiplication of real numbers.

A) True.

B) False.



Multiple Choice

4.2 Subspaces

Definition

Subspace

A subset W of a vector space V is called a subspace of V if W is itself a vector space under the addition and scalar multiplication defined on V .

NOTE. By just being a subset of V , W must already satisfy all 10 axioms except possibly (1), (4), (5), and (6) the others are inherited from V . But (5) is satisfied if (6) is satisfied, since $-u = (-1)u$. This leads to the following theorem:

Theorem

Test For a Subspace

A subset W of a vector space V is a subspace if and only if the following conditions are satisfied:

1. $0 \in W$ (The zero vector of V).
2. If $u, v \in W$, then $u + v \in W$.
3. If c is a scalar and $u \in W$, then $cu \in W$.

Example

The Trivial Subspaces

If V is a nonzero vector space then V has at least two subspaces, namely, V itself and the zero subspace $\{0\}$.

Example

The Subspace of Polynomials P_∞

- Recall that a polynomial is a function that can be written as

$$f = a_0 + a_1x + \cdots + a_kx^k \text{ where } a_0, a_1, \cdots, a_k \text{ are constants.}$$

Clearly,

- the sum of two polynomials is a polynomial, (closed under addition)
 - a constant times a polynomial is a polynomial, (closed under scalar multiplication)
- So, the set of all polynomials is closed under addition and scalar multiplication and hence is a subspace of $F(-\infty, \infty)$. We will denote this space by P_∞ .

Example

The Subspace of Polynomials of degree $\leq n$

Recall that

- The degree of the polynomial is the highest power of its variable with nonzero coefficient. E.g. $3 - 4x^2 - x^4$ has degree 4.
 - the sum of two polynomials cannot have a higher degree than both polynomials.
 - Scalar multiplication cannot increase the degree.
- So, the set of all polynomials of degree n or less is closed under addition and scalar multiplication and hence is a subspace of $F(-\infty, \infty)$. We will denote this space by P_n .

Example

The Symmetric Matrices is a Subspace of $M_{2 \times 2}$

Let W be the set of all 2×2 symmetric matrices. Show that W is a subspace of the vector space $M_{2 \times 2}$, with the standard operations of matrix addition and scalar multiplication.

Solution.

$$(1) 0^T = 0 \Rightarrow 0 \in W$$

$$(2) A_1 \in W, A_2 \in W \Rightarrow (A_1 + A_2)^T = A_1^T + A_2^T = A_1 + A_2 \quad (A_1 + A_2 \in W)$$

$$(3) c \in R, A \in W \Rightarrow (cA)^T = cA^T = cA \quad (cA \in W)$$

The definition of a symmetric matrix A is that $A^T = A$

Note: The same argument shows that in general the set of symmetric $n \times n$ matrices is a subspace of $M_{n \times n}$.

Example

The Singular Matrices is NOT a Subspace of $M_{2 \times 2}$

Let W be the set of all 2×2 singular matrices. Show that W is NOT a subspace of the vector space $M_{2 \times 2}$, with the standard matrix operations.

Solution.

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \in W, \quad B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \in W$$

$$\therefore A + B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I \notin W \quad (W \text{ is not closed under vector addition})$$

$\therefore W$ is not a subspace of $M_{2 \times 2}$

Note: A similar argument shows that in general the set of singular $n \times n$ matrices is NOT a subspace of $M_{n \times n}$.

Exercise:

True or False?

The set of all 2×2 **invertible** matrices is a subspace of $M_{2 \times 2}$.

A) True.

B) False.

 Multiple Choice

Example**The First Quadrant is NOT a Subspace of R^2**

Show that the set $W = \{(x_1, x_2): x_1, x_2 \geq 0\}$ is NOT a subspace of R^2 with the standard.

Solution.

Let $\mathbf{u} = (1, 1) \in W$

$\therefore (-1)\mathbf{u} = (-1)(1, 1) = (-1, -1) \notin W$
(W is not closed under scalar multiplication)

$\therefore W$ is not a subspace of R^2

Example

Identifying Subspaces of R^2

Which of the following two subsets is a subspace of R^2 ?

- (a) The set of points on the line given by $x + 2y = 0$.
- (b) The set of points on the line given by $x + 2y = 1$.

Solution. (a) $W = \{(x, y) \mid x + 2y = 0\} = \{(-2t, t) \mid t \in R\}$

The zero vector $(0,0)$ is on this line for take $t = 0$.

Let $\mathbf{v}_1 = (-2t_1, t_1) \in W$ and $\mathbf{v}_2 = (-2t_2, t_2) \in W$

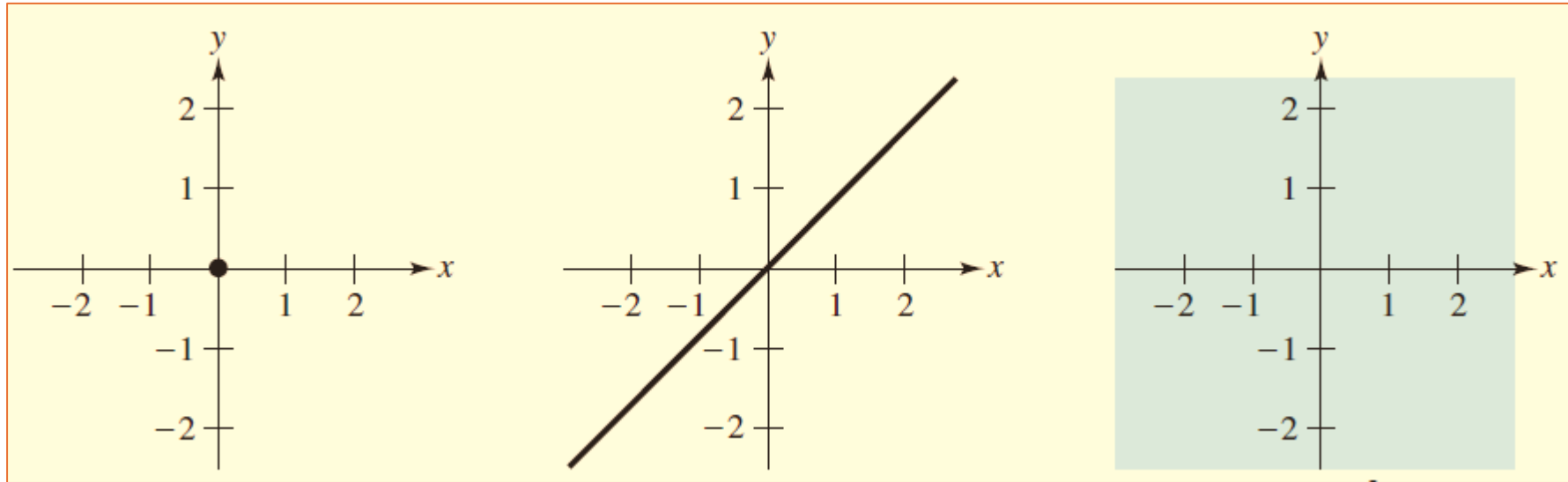
$\therefore \mathbf{v}_1 + \mathbf{v}_2 = (-2(t_1 + t_2), t_1 + t_2) \in W$ (closed under vector addition)

$c\mathbf{v}_1 = (-2(ct_1), ct_1) \in W$ (closed under scalar multiplication) $\therefore W$ is a subspace of R^2

(b) This line clearly doesn't contain the zero vector $(0,0)$, hence NOT a subspace.

Note: We'll see later that solutions of homogeneous linear systems are always subspaces while solutions of nonhomogeneous linear systems are clearly never subspaces, why?

Subspaces of R^2



(1) $\{0\}$.
(trivial subspace)

(2) Lines through
the origin.

(3) R^2 .
(trivial subspace)

Note: We'll see later that solutions of homogeneous linear systems are always subspaces while solutions of nonhomogeneous linear systems are clearly never subspaces, why?

Example

Identifying Subspaces of R^3

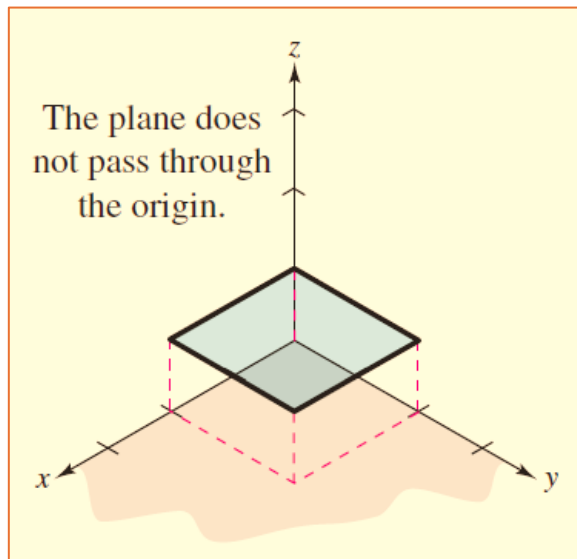
Which of the following two subsets is a subspace of R^3 ?

(a) $W = \{(x_1, x_2, 1) : x_1, x_2 \in R\}$.

(b) $W = \{(x_1, x_1 + x_3, x_3) : x_1, x_3 \in R\}$.

Solution.

(a)



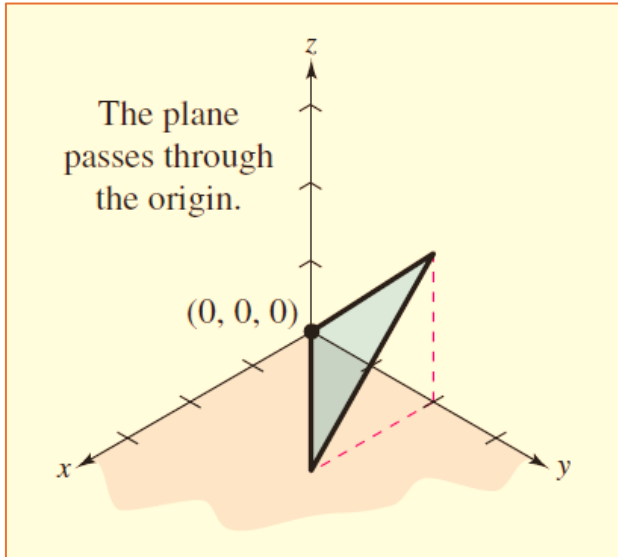
Consider $\mathbf{v} = (0, 0, 1) \in W$

$\therefore (-1)\mathbf{v} = (0, 0, -1) \notin W$

$\therefore W$ is not a subspace of R^3

(Note: the zero vector is not in W)

(b)



Note that the zero vector $(0,0,0)$ is on this set.

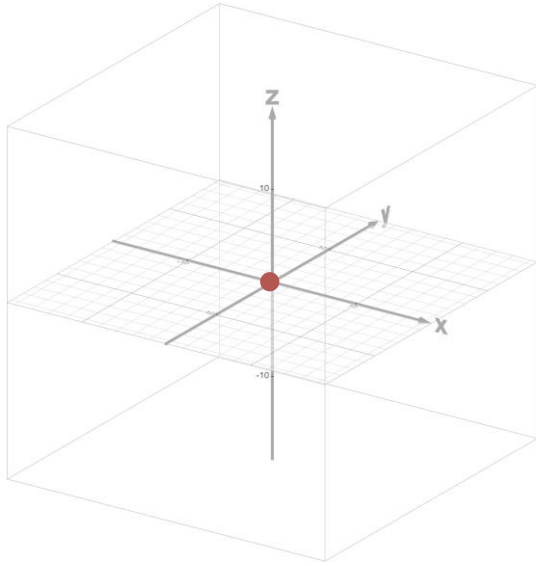
Consider $\mathbf{v} = (v_1, v_1 + v_3, v_3) \in W$ and $\mathbf{u} = (u_1, u_1 + u_3, u_3) \in W$

$$\therefore \mathbf{v} + \mathbf{u} = (v_1 + u_1, (v_1 + u_1) + (v_3 + u_3), v_3 + u_3) \in W$$

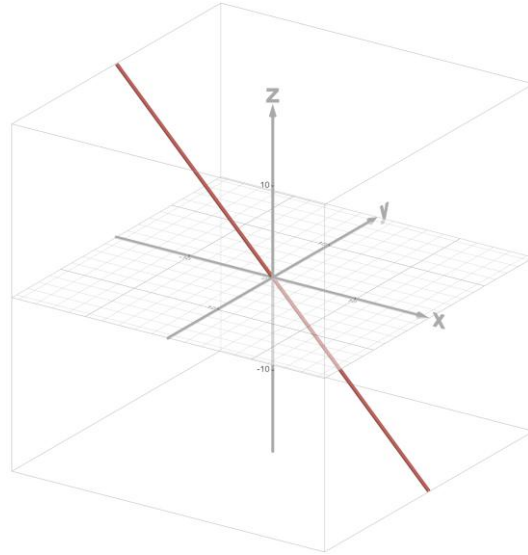
$$c\mathbf{v} = (cv_1, (cv_1) + (cv_3), cv_3) \in W$$

$\therefore W$ is closed under vector addition and scalar multiplication,
so W is a subspace of R^3

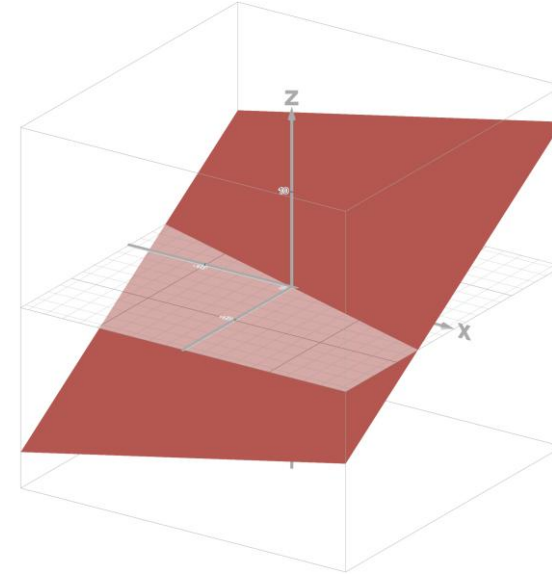
Subspaces of R^3



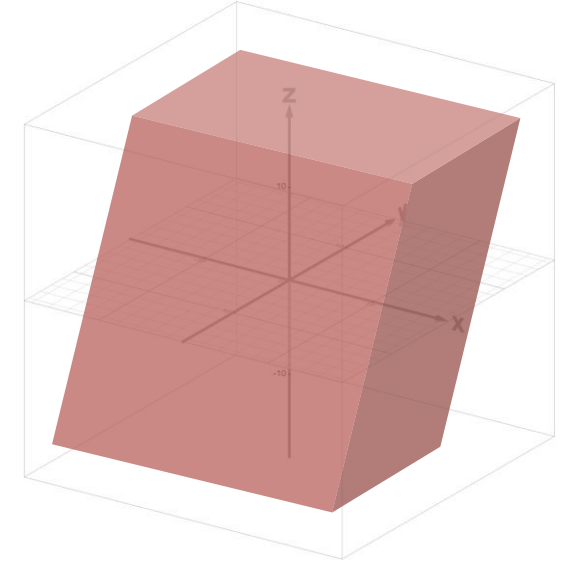
(1) $\{0\}$.
(trivial subspace)



(2) Lines through
the origin.



(3) Planes through
the origin.



(4) R^3 .
(trivial subspace)

Creating Subspaces

Theorem

The Intersection of Subspaces is a Subspace

If V and W are both subspaces of a vector space U , then the intersection of V and W (denoted by $V \cap W$) is also a subspace of U .

Proof. (1) $0 \in V \cap W$, since 0 is in V and W because they are subspaces.

(2) For \mathbf{v}_1 and \mathbf{v}_2 in $V \cap W$, since \mathbf{v}_1 and \mathbf{v}_2 are in V (and W), $\mathbf{v}_1 + \mathbf{v}_2$ is in V (and W).

Therefore, $\mathbf{v}_1 + \mathbf{v}_2$ is in $V \cap W$.

(3) For \mathbf{v}_1 in $V \cap W$, since \mathbf{v}_1 is in V (and W), $c\mathbf{v}_1$ is in V (and W).

Therefore, $c\mathbf{v}_1$ is in $V \cap W$.

Notes:

- The theorem is easily generalized for any finite intersection of subspaces.
- The union of subspaces may NOT be a subspace in general.

Definition

Linear Combination and Span

Let V be a vector space and $S = \{v_1, v_2, \dots, v_k\} \subseteq V$.

- A vector of the form $c_1v_1 + c_2v_2 + \dots + c_kv_k$, where $c_1, c_2, \dots, c_k \in R$ is called a **linear combination** of the v_i s.
- The set all such linear combinations is called the **span of S** and is written as:

$$\text{span}(S) = \{c_1v_1 + c_2v_2 + \dots + c_kv_k : c_1, c_2, \dots, c_k \in R\}.$$

Theorem

$\text{span}(S)$ is a Subspace

If V be a vector space and $S = \{v_1, v_2, \dots, v_k\} \subseteq V$, then

- a) $\text{span}(S)$ is a subspace of V .
- b) $\text{span}(S)$ is the smallest subspace of V containing S , i.e., every other subspace of V containing S must contain $\text{span}(S)$.

Proof. a) First $0 = 0v_1 + 0v_2 + \cdots + 0v_k$, so $0 \in \text{span}(S)$. Consider any two vectors u and v in $\text{span}(S)$, that is,

$$u = c_1v_1 + c_2v_2 + \cdots + c_kv_k \text{ and } v = d_1v_1 + d_2v_2 + \cdots + d_kv_k$$

Then

- $u + v = (c_1 + d_1)v_1 + (c_2 + d_2)v_2 + \cdots + (c_k + d_k)v_k \in \text{span}(S)$, and
- $cu = (cc_1)v_1 + (cc_2)v_2 + \cdots + (cc_k)v_k \in \text{span}(S)$

So, we can conclude that $\text{span}(S)$ is a subspace of V .

b) Let U be another subspace of V containing S . We want to show $\text{span}(S) \subset U$.

Consider any $u \in \text{span}(S)$, i.e., $u = \sum_{i=1}^k c_i v_i$, where $v_i \in S$

U contains $S \Rightarrow v_i \in U$ $\xRightarrow{U \text{ is a subspace}}$ $u = \sum_{i=1}^k c_i v_i \in U$ (because U is closed under vector addition and scalar multiplication)

Since for any vector $\mathbf{u} \in \text{span}(S)$, \mathbf{u} also belongs to U , then $\text{span}(S) \subset U$.

Example

Finding a linear combination

Let $\mathbf{v}_1 = (1,2,3)$ $\mathbf{v}_2 = (0,1,2)$ $\mathbf{v}_3 = (-1,0,1)$. Show that

(a) $\mathbf{w} = (1,1,1)$ is a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$

(b) $\mathbf{w} = (1, -2, 2)$ is not a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$

Solution (a) $\mathbf{w} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3$

$$\begin{aligned} \Rightarrow (1,1,1) &= c_1(1,2,3) + c_2(0,1,2) + c_3(-1,0,1) \\ &= (c_1 - c_3, 2c_1 + c_2, 3c_1 + 2c_2 + c_3) \end{aligned} \Rightarrow \begin{array}{r} c_1 - c_3 = 1 \\ 2c_1 + c_2 = 1 \\ 3c_1 + 2c_2 + c_3 = 1 \end{array} \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 2 & 1 & 0 & 1 \\ 3 & 2 & 1 & 1 \end{array} \right] \xrightarrow{\text{G.-J. E.}} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

\Rightarrow This system has infinitely many solutions. $\Rightarrow \mathbf{w}$ can be expressed as $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3$.

(b) $\mathbf{w} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 2 & 1 & 0 & -2 \\ 3 & 2 & 1 & 2 \end{array} \right] \xrightarrow{\text{G.-J. E.}} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 0 & 1 & 2 & -4 \\ 0 & 0 & 0 & 7 \end{array} \right]$$

\Rightarrow This system has no solution since the third row means $0 \cdot c_1 + 0 \cdot c_2 + 0 \cdot c_3 = 7$

$\Rightarrow \mathbf{w}$ can not be expressed as $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3$

Definition

A Spanning Set For a Vector Space

If V is a vector space and $S \subseteq V$ a subset such that $\text{span}(S) = V$, then S is called a **spanning** set or a **generating** set for V .

Note: Since we know $\text{span}(V) = V$, V is a spanning set for itself. We are interested in small sets that span V .

Example

A Standard Spanning Set For R^3

The set $S = \{(1,0,0), (0,1,0), (0,0,1)\}$ spans R^3 because any vector $\mathbf{u} = (u_1, u_2, u_3)$ in R^3 can be written as $\mathbf{u} = u_1(1,0,0) + u_2(0,1,0) + u_3(0,0,1)$

Example

A Standard Spanning Set For P_2

The set $S = \{1, x, x^2\}$ spans P_2 because any polynomial $p(x) = a + bx + cx^2$ in P_2 can be written as $p(x) = a(1) + b(x) + c(x^2)$.

Example

A Non-Standard Spanning Set For R^3

Show that the set $S = \{(1,2,3), (0,1,2), (-2,0,1)\}$ spans R^3 .

Solution We must show any vector $\mathbf{u} = (u_1, u_2, u_3)$ in R^3 can be expressed as a linear combination of $\mathbf{v}_1 = (1,2,3)$, $\mathbf{v}_2 = (0,1,2)$, and $\mathbf{v}_3 = (-2,0,1)$

$$\text{If } \mathbf{u} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 \quad \Rightarrow \quad \begin{aligned} c_1 - 2c_3 &= u_1 \\ 2c_1 + c_2 &= u_2 \\ 3c_1 + 2c_2 + c_3 &= u_3 \end{aligned}$$

The above problem thus reduces to determine whether this system is consistent for all values of u_1 , u_2 , and u_3 .

$$\therefore |A| = \begin{vmatrix} 1 & 0 & -2 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{vmatrix} \neq 0$$

✧ From a Thm., if A is an invertible matrix, then the system of linear equations $Ax = b$ has a unique solution $x = A^{-1}b$ given any b .

✧ From a Thm., a square matrix A is invertible (nonsingular) if and only if $\det(A) \neq 0$

$\therefore A\mathbf{x} = \mathbf{u}$ has exactly one solution for every $\mathbf{u} \Rightarrow \text{span}(S) = R^3$

Example**A Spanning Set For $M_{2 \times 2}$**

Since $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = a \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + d \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, we have

$$M_{2 \times 2} = \text{span} \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

Example**A Spanning Set For the subspace W of $M_{2 \times 2}$ of Symmetric Matrices**

Since $\begin{bmatrix} a & b \\ b & d \end{bmatrix} = a \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + d \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, we have

$$W = \text{span} \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

Note: Writing a subset of a vector space as a span of a set shows it is a subspace.

Exercise:

Which of the following is a spanning set for the subspace of $M_{2 \times 2}$ of **diagonal matrices**?

- A) $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right\}$
- B) $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\}$
- C) $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right\}$
- D) $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$

 Multiple Choice

Exercise:

Which of the following is a spanning set for the subspace of $M_{2 \times 2}$ of **diagonal matrices**?

- A) $\left\{ \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \right\}$
- B) $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \right\}$
- C) $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right\}$
- D) $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\}$

 Multiple Choice

Theorem Solution Sets of Homogeneous linear Systems are Subspaces of R^n

The solution set of a homogeneous linear system $Ax = 0$ of m equations in n unknowns is a subspace of R^n .

Proof Let W be the solution set of the system. Then $0 \in W$ because $A0 = 0$. Now let $x_1, x_2 \in W$ and $c \in R$. Then $Ax_1 = 0, Ax_2 = 0$ and we have

$$A(x_1 + x_2) = Ax_1 + Ax_2 = 0 + 0 = 0.$$

Also,

$$A(cx_1) = cAx_1 = c0 = 0.$$

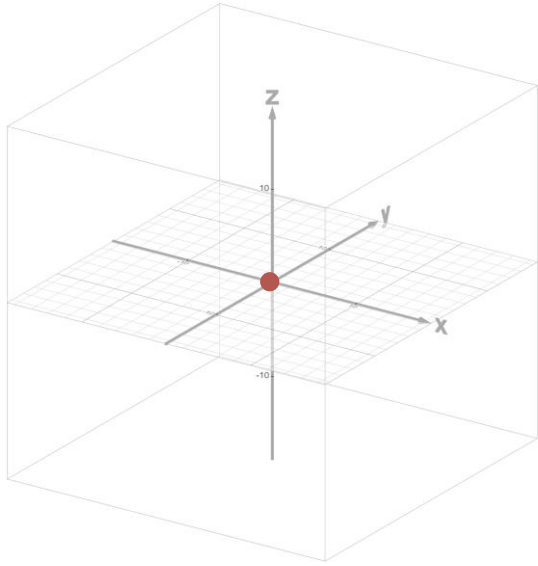
Example Solution Spaces of Homogeneous Systems

$W = \{(x_1, x_2, x_3) \in R^3 : x_1 + 2x_2 + 3x_3 = 0, 4x_1 + 5x_2 + 6x_3 = 0\}$ is a subspace of R^3 because it is the solution set of a homogeneous linear

system $Ax = 0$, where $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$.

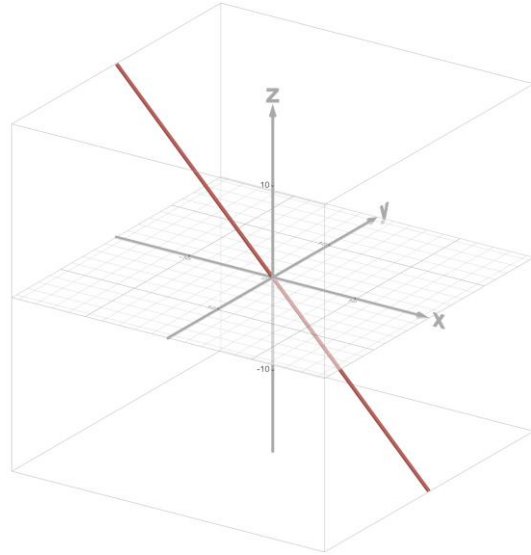
Example

Solution Spaces of Homogeneous Systems



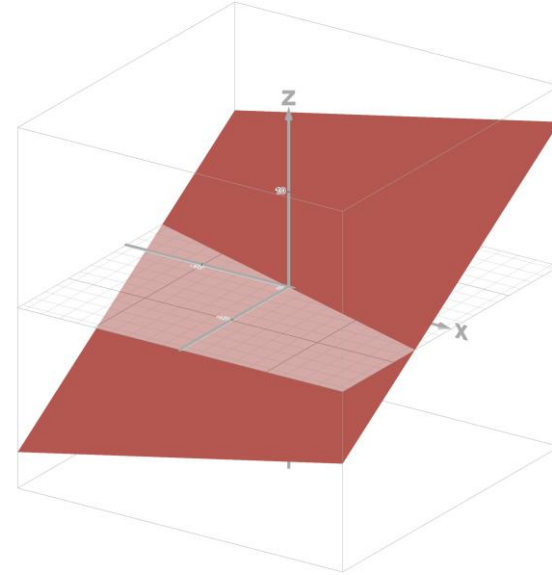
$$(1) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

(trivial subspace)



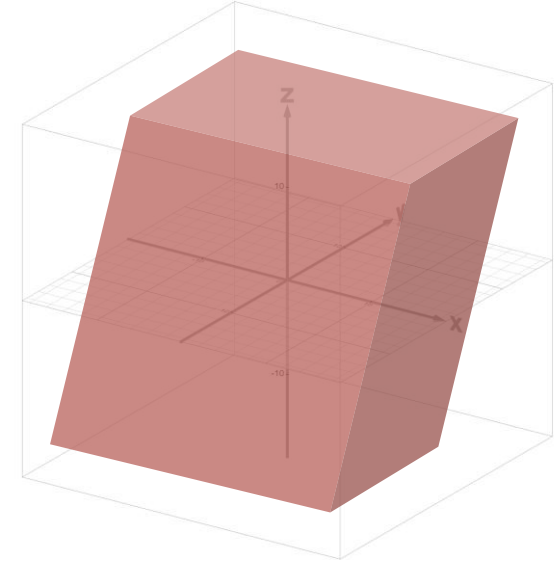
$$(2) \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

Lines through the origin.



$$(3) \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

Planes through the origin.



$$(4) \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

\mathbb{R}^3
(trivial subspace)