4 Vector Spaces

4.1 Real Vector Spaces

4.2 Subspaces

4.1 Real Vector Spaces

Definition Vector Space

Let V be a set on which two operations (addition and scalar multiplication) are defined. **If the following ten axioms are satisfied** for every elements **u**, **v**, and **w** in V and every scalars (real numbers) c and d, then ***V* is called a vector space**, and the **elements** in ***V*** are called **vectors**

Addition:

(1) $\mathbf{u}+\mathbf{v}$ is in V

(2) $\mathbf{u}+\mathbf{v} = \mathbf{v}+\mathbf{u}$

(3) $\mathbf{u}+(\mathbf{v}+\mathbf{w}) = (\mathbf{u}+\mathbf{v})+\mathbf{w}$

(4) V has a zero vector **0** such that for every **u** in V, $\mathbf{u}+\mathbf{0} = \mathbf{u}$

(5) For every **u** in V, there is a vector in V denoted by $-\mathbf{u}$ such that $\mathbf{u}+(-\mathbf{u}) = \mathbf{0}$

Scalar multiplication:

(6) $c\mathbf{u}$ is in V

(7) $c(\mathbf{u}+\mathbf{v}) = c\mathbf{u}+c\mathbf{v}$

(8) $(c+d)\mathbf{u} = c\mathbf{u}+d\mathbf{u}$

(9) $c(d\mathbf{u}) = (cd)\mathbf{u}$

(10) $1(\mathbf{u}) = \mathbf{u}$

Notes A vector space consists of four entities

a set of vectors, a set of real-number scalars, and two operations

V: nonempty set of vectors
c: any scalar

$+(\mathbf{u}, \mathbf{v}) = \mathbf{u} + \mathbf{v}$: vector addition

$\cdot(c, \mathbf{u}) = c\mathbf{u}$: scalar multiplication

$(V, \ +, \ \cdot)$ is called a vector space

The set *V* together with the definitions of vector addition and scalar multiplication satisfying the above ten axioms is called a **vector space**

Example $M_{m\times n}$ is a Vector Space

Proof Recall that the usual matrix addition and scalar multiplication have properties for any $A, B, C \in M_{m\times n}$ and any $s, t \in \mathbb{R}$:

1. $A + B \in M_{m\times n}$, **closed** under addition.
2. $A + B = B + A$, addition is **commutative**
3. $(A + B) + C = A + (B + C)$, addition is **associative**
4. There exists a zero matrix O_{mn}, such that $A + O_{mn} = A$, additive **identity**.
5. There exists a matrix $-A \in M_{m\times n}$ such that $A + (-A) = O_{mn}$, additive **inverse**.

6. $sA \in M_{m\times n}$, **closed** under scalar multiplication.
7. $s(A + B) = sA + sB$, **scalar distribution**.
8. $(s + t)A = sA + tA$ **matrix distribution**.
9. $s(tA) = (st)A$, scalar multiplication is **associative**.
10. $1A = A$, **scalar multiplicative identity**.

Example The Space R^n

An ordered *n*-tuple: a sequence of *n* real numbers $(x_1, x_2, \cdots, x_n)$

R^n-space : the set of all ordered *n*-tuples

n = 1 R^1-space = set of all real numbers
(R^1-space can be represented geometrically by the *x*-axis)

n = 2 R^2-space = set of all ordered pair of real numbers (x_1, x_2)
(R^2-space can be represented geometrically by the *xy*-plane)

n = 3 R^3-space = set of all ordered triple of real numbers (x_1, x_2, x_3)
(R^3-space can be represented geometrically by the *xyz*-space)

For $\mathbf{u} = (u_1, u_2, \cdots, u_n)$, $\mathbf{v} = (v_1, v_2, \cdots, v_n)$ (two vectors in R^n)

Equality: $\mathbf{u} = \mathbf{v}$ if and only if $u_1 = v_1,\ u_2 = v_2,\ \cdots,\ u_n = v_n$

Vector addition (the sum of **u** and **v**): $\mathbf{u} + \mathbf{v} = (u_1 + v_1, u_2 + v_2, \cdots, u_n + v_n)$

Scalar multiplication (the scalar multiple of **u** by c): $c\mathbf{u} = (cu_1, cu_2, \cdots, cu_n)$

Note: This addition and scalar multiplication are called the **standard operations** for R^n.

Zero vector: $\mathbf{0} = (0, 0, ..., 0)$

additive inverse: $-u = (-u_1, -u_2, \cdots, -u_n)$

Notes:

A vector $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ in R^n can be viewed as:

Use comma to separate components

a $1\times n$ row matrix (row vector): $\mathbf{u} = [u_1 \ u_2 \ \cdots \ u_n]$

Use blank space to separate entries

or

a $n\times 1$ column matrix (column vector): $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$

※ Therefore, the operations of matrix addition and scalar multiplication generate the same results as the corresponding vector operations

Vector addition

$$\mathbf{u}+\mathbf{v}=(u_1,u_2,\cdots,u_n)+(v_1,v_2,\cdots,v_n)$$
$$=(u_1+v_1,u_2+v_2,\cdots,u_n+v_n)$$

Scalar multiplication

$$c\mathbf{u}=c(u_1,\ u_2,\cdots,\ u_n)$$
$$=(cu_1,cu_2,\cdots,\ cu_n)$$

Regarded as $1\times n$ row matrix

$$\mathbf{u}+\mathbf{v}=[u_1\ u_2\ \cdots\ u_n]+[v_1\ v_2\ \cdots\ v_n]$$
$$=[u_1+v_1\ u_2+v_2\ \cdots\ u_n+v_n]$$

$$c\mathbf{u}=c[u_1\ u_2\cdots u_n]$$
$$=[cu_1\ cu_2\ \cdots\ cu_n]$$

Regarded as $n\times 1$ column matrix

$$\mathbf{u}+\mathbf{v}=\begin{bmatrix}u_1\\u_2\\\vdots\\u_n\end{bmatrix}+\begin{bmatrix}v_1\\v_2\\\vdots\\v_n\end{bmatrix}=\begin{bmatrix}u_1+v_1\\u_2+v_2\\\vdots\\u_n+v_n\end{bmatrix}$$

$$c\mathbf{u}=c\begin{bmatrix}u_1\\u_2\\\vdots\\u_n\end{bmatrix}=\begin{bmatrix}cu_1\\cu_2\\\vdots\\cu_n\end{bmatrix}$$

Example R^n is a Vector Space

Proof By regarding R^n as row (or column) matrices i.e. $R^n = M_{1\times n}$ (or $R^n = M_{n\times 1}$), the fact that R^n is vector space becomes a special case for that of matrices.

Example The Zero Vector Space

Let V consist of a single object, which we denote by 0, that is $V = \{0\}$. Define

$$0 + 0 = 0, \text{ and } c0 = 0 \text{ for any } c \in R.$$

It is easy to check all 10 axioms are satisfies. V is called the zero vector space.

Example The Vector Space of infinite sequences R^{∞}

- The set of all infinite sequences $u = (u_1, u_2, u_3, \cdots)$ is denoted by R^{∞}.
- $u = (u_1, u_2, u_3, \cdots), v = (v_1, v_2, v_3, \cdots)$ are said to be equal if $u_i = v_i, \forall i \geq 1$.
- For $u, v \in R^{\infty}$ and $c \in R$, addition and scalar multiplication are defined by:

$$u + v = (u_1, u_2, u_3, \cdots) + (v_1, v_2, v_3, \cdots) = (u_1 + v_1, u_{12} + v_2, u_3 + v_3, \cdots)$$

$$cu = (cu_1, cu_2, cu_3, \cdots)$$

- This makes R^{∞} into a vector space, where

$$0 = (0,0,0, \cdots) \text{ and } -u = (-u_1, -u_2, -u_3, \cdots)$$

Example The Vector Space of real Valued functions $F(D)$

- Let $D \subseteq R$. The set of all real valued function $f: D \to R$ is denoted by $F(D)$.
- $f, g \in F(D)$ are said to be equal if $f(x) = g(x), \forall x \in D$.
- For $f, g \in F(D)$ and $c \in R$, addition and scalar multiplication are defined by:

$$(f + g)(x) = f(x) + g(x)$$

$$(cf)(x) = cf(x)$$

- This together with the properties of addition on R makes $F(D)$ into a vector space, where 0 and $-f$ are the functions:

$$0(x) = 0 \text{ and } (-f)(x) = -f(x)$$

Example A Set That Is NOT a Vector Space

If $V = \mathbb{Z}$ is the set of integers, with addition and scalar multiplication. Then V is not a vector space since

$$1 \in V, \text{ and } \tfrac{1}{2} \text{ is a real-number scalar}$$

$$\underset{\text{scalar}}{(\tfrac{1}{2})}\underset{\text{integer}}{(1)} = \underset{\text{noninteger}}{\tfrac{1}{2}} \notin V \quad \text{(it is not closed under scalar multiplication)}$$

Example A Set That Is NOT a Vector Space

If $V = R^2$, with addition and scalar multiplication:

$$(u_1, u_2) + (v_1, v_2) = (u_1 + v_1, u_2 + v_2) \qquad c(u_1, u_2) = (cu_1, 0) \quad \text{(nonstandard definition)}$$

The first nine axioms of the definition of a vector space are satisfied (check it), but NOT tenth axiom. Since for example, $1(1,1) = (1,0) \neq (1,1)$.

Theorem Properties of additive identity and additive inverse

Let v be any element of a vector space V, and let c be any scalar. Then

$$(1)\quad 0\mathbf{v} = \mathbf{0}$$

$$(2)\quad c\mathbf{0} = \mathbf{0}$$

$$(3)\quad \text{If } c\mathbf{v} = \mathbf{0},\quad \text{either } c = 0 \quad \text{or} \quad \mathbf{v} = \mathbf{0}$$

$$(4)\quad (-1)\mathbf{v} = -\mathbf{v}$$ (the additive inverse of $\mathbf{v}$ equals $((-1)\mathbf{v})$)

Proof.

$$(1)\ 0\text{v} = (0+0)\text{v} \overset{(8)}{=} 0\text{v} + 0\text{v} \overset{\text{add}(-0v)}{\Rightarrow} 0\text{v} + (-0\text{v}) = (0\text{v} + 0\text{v}) + (-0\text{v}) \overset{(5,3)}{\Rightarrow} 0 = 0\text{v}$$

$$(2)\qquad c\mathbf{0} \overset{(4)}{=} c(\mathbf{0}+\mathbf{0}) \overset{(7)}{=} c\mathbf{0} + c\mathbf{0}$$

$$\Rightarrow c\mathbf{0} + (-c\mathbf{0}) = (c\mathbf{0} + c\mathbf{0}) + (-c\mathbf{0})$$ (add $(-c\mathbf{0})$ to both sides)

$$\overset{(3)}{\Rightarrow} c\mathbf{0} + (-c\mathbf{0}) = c\mathbf{0} + [c\mathbf{0} + (-c\mathbf{0})]$$

$$\overset{(5)}{\Rightarrow} \mathbf{0} = c\mathbf{0} + \mathbf{0} \quad \overset{(4)}{\Rightarrow} \mathbf{0} = c\mathbf{0}$$

(3) Prove by contradiction: Suppose that $c\mathbf{v} = \mathbf{0}$, but $c \neq 0$ and $\mathbf{v} \neq \mathbf{0}$

$$\mathbf{v} \overset{(10)}{=} 1\mathbf{v} = \left(\frac{1}{c}c\right)\mathbf{v} \overset{(9)}{=} \frac{1}{c}(c\mathbf{v}) = \frac{1}{c}(\mathbf{0}) = \mathbf{0}$$ (By the second property, $c\mathbf{0} = \mathbf{0}$)

$\Rightarrow \rightarrow\leftarrow \quad \Rightarrow$ if $c\mathbf{v}=\mathbf{0}$, either $c = 0$ or $\mathbf{v} = \mathbf{0}$

(4) $$0\mathbf{v} = (1+(-1))\mathbf{v} \overset{(8)}{=} 1\mathbf{v} + (-1)\mathbf{v}$$

$\Rightarrow \mathbf{0} = \mathbf{v} + (-1)\mathbf{v}$ (By the first property, $0\mathbf{v} = \mathbf{0}$)

$\overset{(5)}{\Rightarrow} (-1)\mathbf{v} = -\mathbf{v}$ (By comparing with Axiom (5), $(-1)\mathbf{v}$ is the additive inverse of $\mathbf{v}$)

4.2 Subspaces

Definition Subspace

A subset W of a vector space V is called a subspace of V if W is itself a vector space under the addition and scalar multiplication defined on V.

NOTE. By just being a subset of V, W must already satisfy all 10 axioms except possibly (1) ,(4), (5), and (6) the others are inherited from V. But (5) is satisfied if (6) is satisfied, since $-u = (-1)u$. This leads to the following theorem:

Theorem Test For a Subspace

A subset W of a vector space V is a subspace if and only if the following conditions are satisfied:

1. $0 \in W$ (The zero vector of V).
2. If $u, v \in W$, then $u + v \in W$.
3. If c is a scalar and $u \in W$, then $cu \in W$.

Example The Trivial Subspaces

If V is a nonzero vector space then V has at least two subspaces, namely , V itself and the zero subspace $\{0\}$.

Example The Subspace of Polynomials P_∞

- Recall that a polynomial is a function that can be written as

$$f = a_0 + a_1 x + \cdots + a_k x^k \text{ where } a_0, a_1, \cdots, a_k \text{ are constants.}$$

Clearly,

- the sum of two polynomials is a polynomial, (closed under addition)
- a constant times a polynomial is a polynomial, (closed under scalar multiplication)

➢ So, the set of all polynomials is closed under addition and scalar multiplication and hence is a subspace of $F(-\infty, \infty)$. We will denote this space by P_∞.

Example The Subspace of Polynomials of degree $\leq n$

Recall that

- The degree of the polynomial is the highest power of its variable with nonzero coefficient. E.g. $3 - 4x^2 - x^4$ has degree 4.

- the sum of two polynomials cannot have a higher degree than both polynomials.
- Scalar multiplication cannot increase the degree.

➢ So, the set of all polynomials of degree n or less is closed under addition and scalar multiplication and hence is a subspace of $F(-\infty, \infty)$. We will denote this space by P_n.

Example The Symmetric Matrices is a Subspace of $M_{2\times 2}$

Let W be the set of all 2×2 symmetric matrices. Show that W is a subspace of the vector space $M_{2\times 2}$, with the standard operations of matrix addition and scalar multiplication.

Solution.

$$(1)\ 0^T = 0 \Rightarrow \ 0 \in W$$

(2) $A_1 \in W, A_2 \in W \Rightarrow (A_1 + A_2)^T = A_1^T + A_2^T = A_1 + A_2 \quad (A_1 + A_2 \in W)$

(3) $c \in R,\ A \in W \Rightarrow (cA)^T = cA^T = cA \quad (cA \in W)$

The definition of a symmetric matrix A is that $A^T = A$

Note: The same argument shows that in general the set of symmetric $n \times n$ matrices is a subspace of $M_{n\times n}$.

Example The Singular Matrices is NOT a Subspace of $M_{2\times2}$

Let W be the set of all 2×2 singular matrices. Show that W is NOT a subspace of the vector space $M_{2\times2}$, with the standard matrix operations.

Solution.

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \in W, \; B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \in W$$

$$\because A + B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I \notin W \quad (W \text{ is not closed under vector addition})$$

$$\therefore W \text{ is not a subspace of } M_{2\times2}$$

Note: A similar argument shows that in general the set of singular $n \times n$ matrices is NOT a subspace of $M_{n\times n}$.

Example The First Quadrant is NOT a Subspace of R^2

Show that the set $W = \{(x_1, x_2): x_1, x_2 \geq 0\}$ is NOT a subspace of R^2 with the standard.

Solution.

$$\text{Let } \mathbf{u} = (1,\ 1) \in W$$

$$\because (-1)\mathbf{u} = (-1)(1, 1) = (-1, -1) \notin W$$

(W is not closed under scalar multiplication)

$$\therefore W \text{ is not a subspace of } R^2$$

Example Identifying Subspaces of R^2

Which of the following two subsets is a subspace of R^2?

(a) The set of points on the line given by $x + 2y = 0$.

(b) The set of points on the line given by $x + 2y = 1$.

Solution. (a) $W = \{(x, y) \mid x + 2y = 0\} = \{(-2t, t) \mid t \in R\}$

The zero vector (0,0) is on this line for take $t = 0$.

Let $\mathbf{v}_1 = (-2t_1, t_1) \in W$ and $\mathbf{v}_2 = (-2t_2, t_2) \in W$

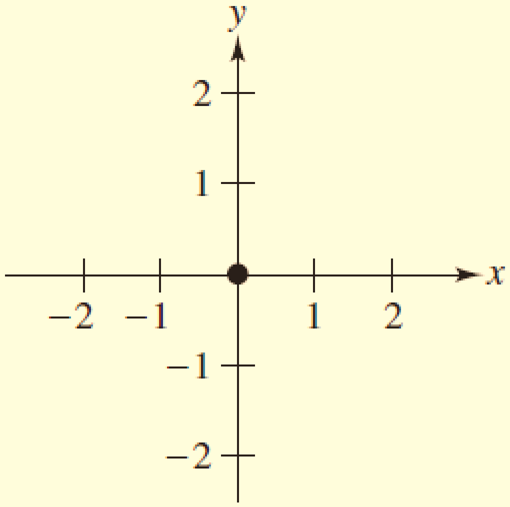
$\because \mathbf{v}_1 + \mathbf{v}_2 = (-2(t_1 + t_2), t_1 + t_2) \in W$ (closed under vector addition)

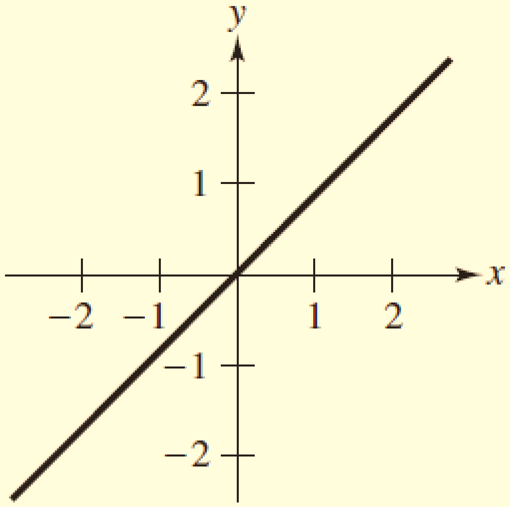
$c\mathbf{v}_1 = (-2(ct_1), ct_1) \in W$ (closed under scalar multiplication) $\therefore W$ is a subspace of R^2

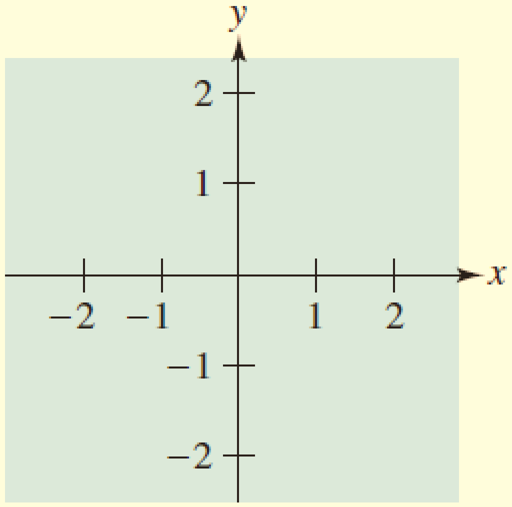
(b) This line clearly doesn't contain the zero vector (0,0), hence NOT a subspace.

Note: We'll see later that solutions of homogeneous linear systems are always subspaces while solutions of nonhomogeneous linear systems are clearly never subspaces, why?

Subspaces of R^2







(1) $\{0\}$.
(trivial subspace)

(2) Lines through the origin.

(3) R^2.
(trivial subspace)

Note: We'll see later that solutions of homogeneous linear systems are always subspaces while solutions of nonhomogeneous linear systems are clearly never subspaces, why?

Example Identifying Subspaces of R^3

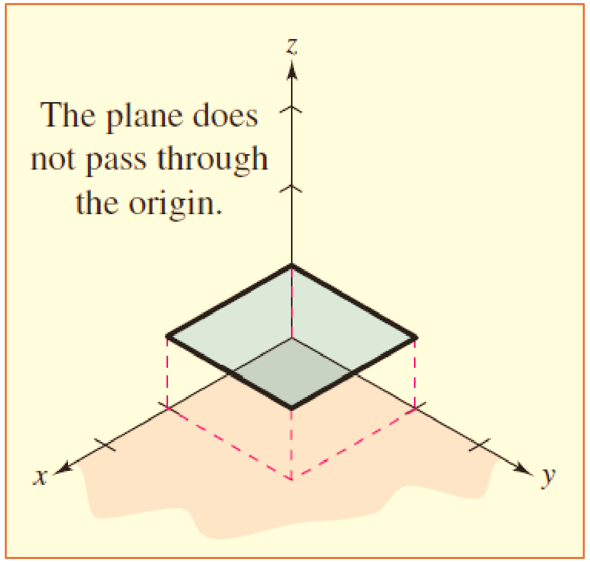
Which of the following two subsets is a subspace of R^3?

(a) $W = \{(x_1, x_2, 1): x_1, x_2 \in R\}$.

(b) $W = \{(x_1, x_1 + x_3, x_3): x_1, x_3 \in R\}$.

Solution.

(a)



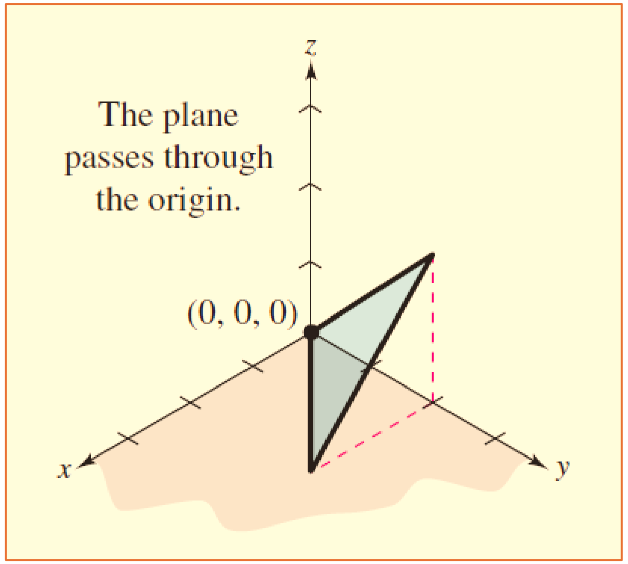
Consider $\mathbf{v} = (0, 0, 1) \in W$

$\because (-1)\mathbf{v} = (0, 0, -1) \notin W$

$\therefore W$ is not a subspace of R^3

(Note: the zero vector is not in W)

(b)



Note that the zero vector (0,0,0) is on this set.

Consider $\mathbf{v} = (v_1, v_1 + v_3, v_3) \in W$ and $\mathbf{u} = (u_1, u_1 + u_3, u_3) \in W$

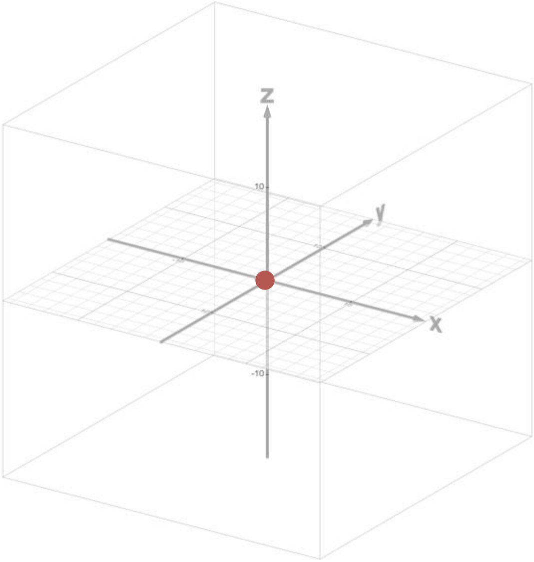
$\because \mathbf{v} + \mathbf{u} = \left(v_1 + u_1, \left(v_1 + u_1\right) + \left(v_3 + u_3\right), v_3 + u_3\right) \in W$

$c\mathbf{v} = \left(cv_1, \left(cv_1\right) + \left(cv_3\right), cv_3\right) \in W$

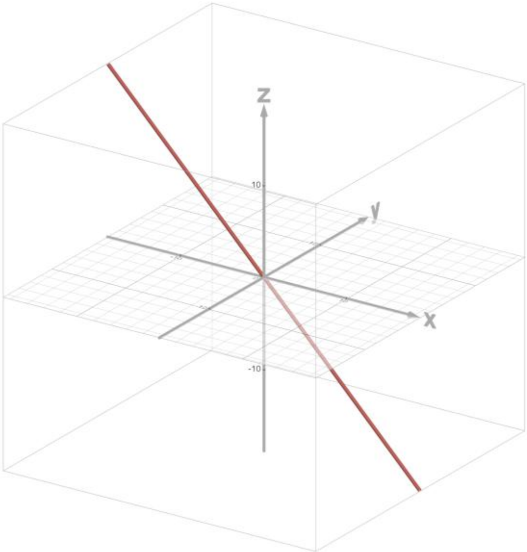
$\therefore W$ is closed under vector addition and scalar multiplication,

so W is a subspace of R^3

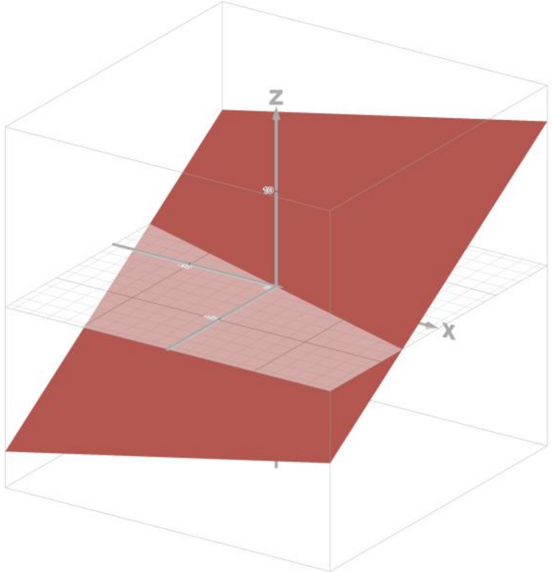
Subspaces of R^3



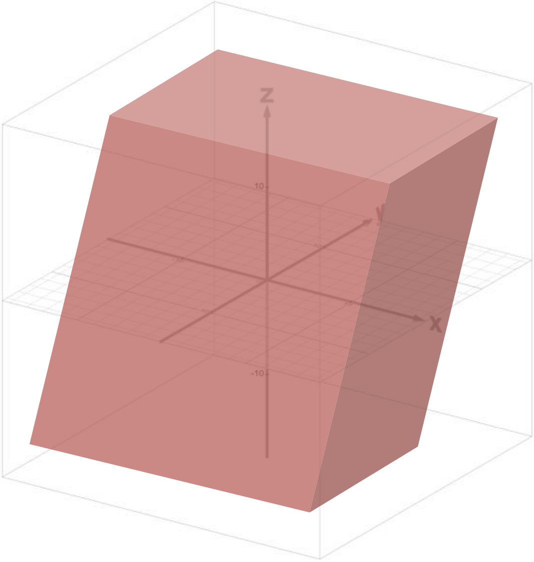
(1) $\{0\}$.
(trivial subspace)



(2) Lines through the origin.



(3) Planes through the origin.



(4) R^3.
(trivial subspace)

Creating Subspaces

Theorem The Intersection of Subspaces is a Subspace

If V and W are both subspaces of a vector space U, then the intersection of V and W (denoted by $V \cap W$) is also a subspace of U.

Proof. (1) $0 \in V \cap W$, since 0 is in V and W because they are subspaces.

(2) For $\mathbf{v}_1$ and $\mathbf{v}_2$ in $V \cap W$, since $\mathbf{v}_1$ and $\mathbf{v}_2$ are in V (and W), $\mathbf{v}_1 + \mathbf{v}_2$ is in V (and W). Therefore, $\mathbf{v}_1 + \mathbf{v}_2$ is in $V \cap W$.

(3) For $\mathbf{v}_1$ in $V \cap W$, since $\mathbf{v}_1$ is in V (and W), $c\mathbf{v}_1$ is in V (and W). Therefore, $c\mathbf{v}_1$ is in $V \cap W$.

Notes:

- The theorem is easily generalized for any finite intersection of subspaces.
- The union of subspaces may NOT be a subspace in general.

Definition Linear Combination and Span

Let V be a vector space and $S = \{v_1, v_2, \cdots, v_k\} \subseteq V$.

- A vector of the form $c_1v_1 + c_2v_2 + \cdots + c_kv_k$, where $c_1, c_2, \dots, c_k \in R$ is called a **linear combination** of the $v_i s$.
- The set all such linear combinations is called the **span of S** and is written as:

$$\text{span(S)} = \{c_1v_1 + c_2v_2 + \cdots + c_kv_k : c_1, c_2, \dots, c_k \in R\}.$$

Theorem $span(S)$ is a Subspace

If V be a vector space and $S = \{v_1, v_2, \cdots, v_k\} \subseteq V$, then

a) $span(S)$ is a subspace of V.
b) $span(S)$ is the smallest subspace of V containing S, , i.e., every other subspace of V containing S must contain $span(S)$.

Proof. a) First $0 = 0v_1 + 0v_2 + \cdots + 0v_k, so\ 0 \in span(S)$. Consider any two vectors u and v in span(S), that is,

$$u = c_1v_1 + c_2v_2 + \cdots + c_kv_k \text{ and } v = d_1v_1 + d_2v_2 + \cdots + d_kv_k$$

Then

- $u + v = (c_1 + d_1)v_1 + (c_2 + d_2)v_2 + \cdots + (c_k + d_k)v_k \in span(S),$ and
- $cu = (cc_1)v_1 + (cc_2)v_2 + \cdots + (cc_k)v_k \in span(S)$

So, we can conclude that span(S) is a subspace of V.

b) Let U be another subspace of V containing S. We want to show span$(S) \subset U$.

Consider any $u \in span(S)$, i.e., $u = \sum_{i=1}^{k} c_iv_i$, where $v_i \in S$

$$U \text{ contains } S \Rightarrow v_i \in U \overset{U \text{ is a subspace}}{\Rightarrow} u = \sum_{i=1}^{k} c_iv_i \in U$$

(because U is closed under vector addition and scalar multiplication)

Since for any vector $\mathbf{u} \in$ span(S), $\mathbf{u}$ also belongs to U, then span$(S) \subset U$.

Example Finding a linear combination

Let $\mathbf{v}_1 = (1,2,3)$ $\mathbf{v}_2 = (0,1,2)$ $\mathbf{v}_3 = (-1,0,1)$. Show that

(a) $\mathbf{w} = (1,1,1)$ is a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$

(b) $\mathbf{w} = (1,-2,2)$ is not a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$

Solution (a) $\mathbf{w} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3$

$$\Rightarrow (1,1,1) = c_1(1,2,3) + c_2(0,1,2) + c_3(-1,0,1)$$
$$= (c_1 - c_3, 2c_1 + c_2, 3c_1 + 2c_2 + c_3)$$

$$\Rightarrow \begin{aligned} c_1 \quad\quad - c_3 &= 1 \\ 2c_1 + c_2 \quad\quad &= 1 \\ 3c_1 + 2c_2 + c_3 &= 1 \end{aligned} \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 2 & 1 & 0 & 1 \\ 3 & 2 & 1 & 1 \end{array}\right] \xrightarrow{\text{G.-J. E.}} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\Rightarrow$ This system has infinitely many solutions. $\Rightarrow$ $\mathbf{w}$ can be expressed as $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3$.

(b) $\mathbf{w} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 2 & 1 & 0 & -2 \\ 3 & 2 & 1 & 2 \end{array}\right] \xrightarrow{\text{G.-J. E.}} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 0 & 1 & 2 & -4 \\ 0 & 0 & 0 & 7 \end{array}\right]$$

$\Rightarrow$ This system has no solution since the third row means $0 \cdot c_1 + 0 \cdot c_2 + 0 \cdot c_3 = 7$

$\Rightarrow$ $\mathbf{w}$ can not be expressed as $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3$

Definition — A Spanning Set For a Vector Space

If V is a vector space and $S \subseteq V$ a subset such that $span(S) = V$, then S is called a **spanning** set or a **generating** set for V.

Note: Since we know $span(V) = V$, V is a spanning set for itself. We are interested in small sets that span V.

Example — A Standard Spanning Set For R^3

The set $S = \{(1,0,0), (0,1,0), (0,0,1)\}$ spans R^3 because any vector $\mathbf{u} = (u_1, u_2, u_3)$ in R^3 can be written as $\mathbf{u} = u_1(1,0,0) + u_2(0,1,0) + u_3(0,0,1)$

Example — A Standard Spanning Set For P_2

The set $S = \{1, x, x^2\}$ spans P_2 because any polynomial $p(x) = a + bx + cx^2$ in P_2 can be written as $p(x) = a(1) + b(x) + c(x^2)$.

Example — A Non-Standard Spanning Set For R^3

Show that the set $S = \{(1,2,3), (0,1,2), (-2,0,1)\}$ spans R^3.

Solution We must show any vector $\mathbf{u} = (u_1, u_2, u_3)$ in R^3 can be expressed as a linear combination of $\mathbf{v}_1 = (1,2,3)$, $\mathbf{v}_2 = (0,1,2)$, and $\mathbf{v}_3 = (-2,0,1)$

If $\mathbf{u} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 \quad \Rightarrow$

$$\begin{aligned} c_1 \qquad\quad - 2c_3 &= u_1 \\ 2c_1 + \quad c_2 \qquad\quad &= u_2 \\ 3c_1 + 2c_2 + c_3 &= u_3 \end{aligned}$$

The above problem thus reduces to determine whether this system is consistent for all values of u_1, u_2, and u_3.

$$\because |A| = \begin{vmatrix} 1 & 0 & -2 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{vmatrix} \neq 0$$

※ From a Thm., if A is an invertible matrix, then the system of linear equations $Ax = b$ has a unique solution $x = A^{-1}b$ given any b.

※ From a Thm., a square matrix A is invertible (nonsingular) if and only if $\det(A)$ 0

$\therefore A\mathbf{x} = \mathbf{u}$ has exactly one solution for every $\mathbf{u} \Rightarrow \text{span}(S) = R^3$

Example A Spanning Set For $M_{2\times 2}$

Since $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + d\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, we have

$$M_{2\times 2} = span\left\{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right\}$$

Example A Spanning Set For the subspace W of $M_{2\times 2}$ of Symmetric Matrices

Since $\begin{bmatrix} a & b \\ b & d \end{bmatrix} = a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + d\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, we have

$$W = span\left\{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right\}$$

Note: Writing a subset of a vector space as a span of a set shows it is a subspace.

Theorem Solution Sets of Homogeneous linear Systems are Subspaces of R^n

The solution set of a homogeneous linear system $Ax = 0$ of m equations in n unknowns is a subspace of R^n.

Proof Let W be the solution set of the system. Then $0 \in W$ because $A0 = 0$. Now let $x_1, x_2 \in W$ and $c \in R$. Then $Ax_1 = 0, Ax_2 = 0$ and we have

$$A(x_1 + x_2) = Ax_1 + Ax_2 = 0 + 0 = 0.$$

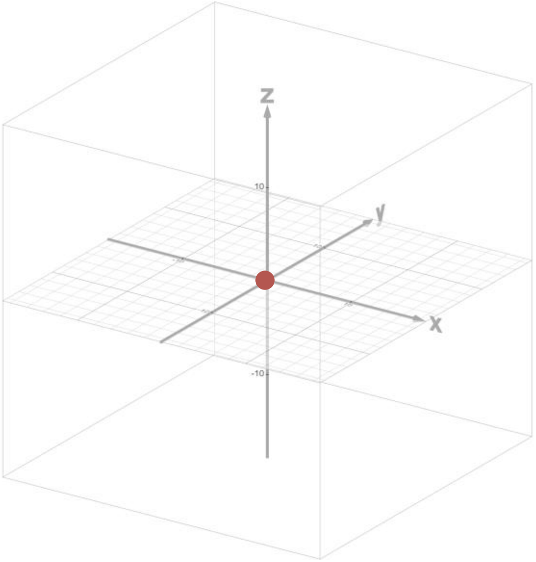
Also,

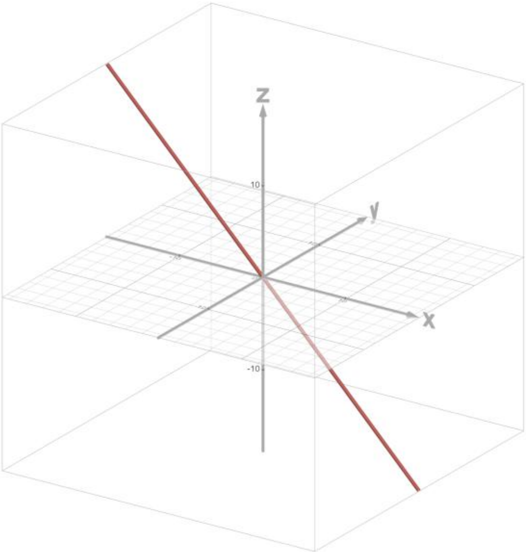
$$A(cx_1) = cAx_1 = c0 = 0.$$

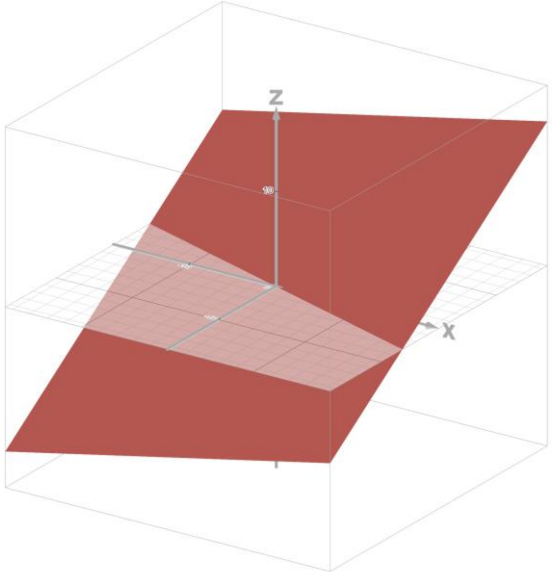
Example Solution Spaces of Homogeneous Systems

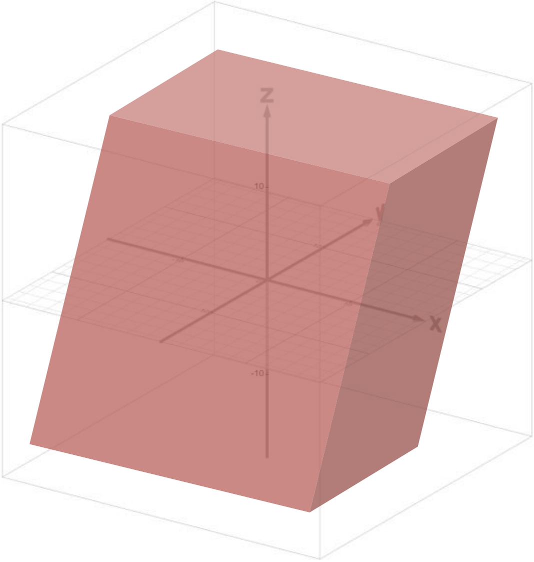
$W = \{(x_1, x_2, x_3) \in R^3 : x_1 + 2x_2 + 3x_3 = 0,\ 4x_1 + 5x_2 + 6x_3 = 0\}$ is a subspace of R^3 because it is the solution set of a homogeneous linear system $Ax = 0$, where $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$.

Example Solution Spaces of Homogeneous Systems









(1) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$

(trivial subspace)

(2) $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$

Lines through the origin.

(3) $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$

Planes through the origin.

(4) $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$

R^3
(trivial subspace)