UNIT 3A

General Vector Spaces

Unit 3A · Sections 4.1 – 4.2

MATH 244 — Linear Algebra
Elementary Linear Algebra, Anton & Rorres, 11th edition
King Saud University · Department of Mathematics

LEARNING OUTCOMES

What you should be able to do

Section 4.1

- State the ten vector space axioms and name the four ingredients of a vector space: a set, the scalars, an addition and a scalar multiplication.
- Show that a set with two operations is a vector space — or that it is not, with one specific counterexample to one axiom.
- Work in $\mathbb{R}^n$, M_{mn}, $\mathbb{R}^\infty$ and $F(-\infty, \infty)$, and find the zero vector and the negatives in any vector space.
- Prove $0\mathbf{u} = \mathbf{0}$, $k\mathbf{0} = \mathbf{0}$ and $(-1)\mathbf{u} = -\mathbf{u}$ from the axioms.

Section 4.2

- Decide whether a subset is a subspace: zero vector first, then the two closure conditions.
- List the subspaces of $\mathbb{R}^2$ and $\mathbb{R}^3$; recognise the standard subspaces of M_{nn} and of the function spaces.
- Decide whether a vector is a linear combination of given vectors, and whether a set spans $\mathbb{R}^n$, by solving a linear system.
- Use spans and solution spaces of $A\mathbf{x} = \mathbf{0}$ to produce subspaces without checking a single axiom.

SECTION 4.1

Real Vector Spaces

GOAL Turn the rules that vectors in $\mathbb{R}^n$ obey into axioms. Anything that obeys them — matrices, functions, sequences — then behaves like a vector, and every theorem proved from the axioms holds for all of them at once.

1. One set of rules, many kinds of objects
2. The ten vector space axioms
3. Standard examples: M_{mn}, $\mathbb{R}^n$, $\mathbb{R}^\infty$, $F(-\infty, \infty)$
4. Sets that are NOT vector spaces
5. An unusual vector space
6. First consequences of the axioms

One set of rules, many kinds of objects

- Add and scale three very different kinds of object:

$$(2,-1,3)+(1,4,0)=(3,3,3), \qquad 2\,(2,-1,3)=(4,-2,6)$$

$$\begin{bmatrix}1 & 2\\ 0 & 3\end{bmatrix}+\begin{bmatrix}-1 & 0\\ 4 & 1\end{bmatrix}=\begin{bmatrix}0 & 2\\ 4 & 4\end{bmatrix}, \qquad 2\begin{bmatrix}1 & 2\\ 0 & 3\end{bmatrix}=\begin{bmatrix}2 & 4\\ 0 & 6\end{bmatrix}$$

$$\left(1+2x-x^2\right)+\left(3-x+4x^2\right)=4+x+3x^2, \qquad 2\left(1+2x-x^2\right)=2+4x-2x^2$$

- Each time the same rules hold: the order of a sum does not matter, there is a zero, every object has a negative, scalars distribute over sums, …

> **IDEA** Collect these rules as AXIOMS. A set whose addition and scalar multiplication obey them is a VECTOR SPACE, and its elements are VECTORS. A theorem proved from the axioms then holds for n-tuples, matrices, polynomials and functions all at once.

DEFINITION 1

The vector space axioms

Let V be a nonempty set with two operations: an ADDITION $\mathbf{u}+\mathbf{v}$ and a SCALAR MULTIPLICATION $k\mathbf{u}$ by real numbers k. V is a VECTOR SPACE, and its objects are VECTORS, if for all $\mathbf{u},\mathbf{v},\mathbf{w}$ in V and all scalars k, m:

Addition

1. $\mathbf{u}+\mathbf{v}$ is in V (closure)
2. $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$
3. $\mathbf{u}+(\mathbf{v}+\mathbf{w})=(\mathbf{u}+\mathbf{v})+\mathbf{w}$
4. there is a ZERO VECTOR $\mathbf{0}$ in V with $\mathbf{0}+\mathbf{u}=\mathbf{u}+\mathbf{0}=\mathbf{u}$
5. each $\mathbf{u}$ has a NEGATIVE $-\mathbf{u}$ in V with $\mathbf{u}+(-\mathbf{u})=(-\mathbf{u})+\mathbf{u}=\mathbf{0}$

Scalar multiplication

6. $k\mathbf{u}$ is in V (closure)
7. $k(\mathbf{u}+\mathbf{v})=k\mathbf{u}+k\mathbf{v}$
8. $(k+m)\mathbf{u}=k\mathbf{u}+m\mathbf{u}$
9. $k(m\mathbf{u})=(km)\mathbf{u}$
10. $1\mathbf{u}=\mathbf{u}$

A vector space has FOUR ingredients: a set V, the scalars $\mathbb{R}$, an addition and a scalar multiplication. Change an operation and the answer may change — Example 8 will show it.

PROCEDURE

Showing that a set is — or is not — a vector space

Step 1. Identify the set V of objects that will be the vectors.

Step 2. Identify the addition and the scalar multiplication on V.

Step 3. Verify Axioms 1 and 6, CLOSURE: adding two vectors of V, or multiplying one by a scalar, gives a vector of V.

Step 4. Confirm Axioms 2, 3, 4, 5, 7, 8, 9 and 10.

> To show that a set is NOT a vector space, one failing axiom is enough. Showing that an axiom HOLDS needs an argument for all vectors and scalars; showing that it FAILS needs one specific example.

EXAMPLES 4 AND 5

M_{mn}, the $m \times n$ matrices, is a vector space

With the usual matrix addition and scalar multiplication, every axiom is a rule of matrix arithmetic we already know:

	The axiom in M_{mn}	Why it holds
1	$A + B$ is again $m \times n$	definition of matrix addition
2	$A + B = B + A$	Theorem 1.4.1(a)
3	$A + (B + C) = (A + B) + C$	Theorem 1.4.1(b)
4	$A + O = O + A = A$	zero vector: the zero matrix O — Theorem 1.4.2(a)
5	$A + (-A) = (-A) + A = O$	negative: $-A$ — Theorem 1.4.2(c)
6	kA is again $m \times n$	definition of a scalar multiple
7	$k(A + B) = kA + kB$	Theorem 1.4.1(h)
8	$(k + m)A = kA + mA$	Theorem 1.4.1(j)
9	$k(mA) = (km)A$	Theorem 1.4.1(l)
10	$1A = A$	definition of a scalar multiple

EXAMPLE 2

The space $\mathbb{R}^n$

An ordered n-tuple is a list $(u_1, u_2, \dots, u_n)$ of n real numbers, and $\mathbb{R}^n = \{(u_1, u_2, \dots, u_n) : u_1, u_2, \dots, u_n \in \mathbb{R}\}$ is the set of all of them. $\mathbb{R}^1$ is the number line, $\mathbb{R}^2$ the plane, $\mathbb{R}^3$ space.

For $\mathbf{u} = (u_1, \dots, u_n)$, $\mathbf{v} = (v_1, \dots, v_n)$ and a scalar k, the STANDARD OPERATIONS are

$$\mathbf{u} + \mathbf{v} = (u_1 + v_1,\ u_2 + v_2,\ \dots,\ u_n + v_n), \qquad k\mathbf{u} = (ku_1,\ ku_2,\ \dots,\ ku_n)$$

$\mathbf{u} = \mathbf{v}$ means $u_1 = v_1,\ \dots,\ u_n = v_n$. The zero vector is $\mathbf{0} = (0, 0, \dots, 0)$, and $-\mathbf{u} = (-u_1, \dots, -u_n)$.

- Write $\mathbf{u}$ as a $1 \times n$ row matrix $[\,u_1 \ \ u_2 \ \cdots \ u_n\,]$ (spaces between entries instead of commas), or as an $n \times 1$ column: the standard operations are then exactly the matrix operations.

> So $\mathbb{R}^n$ is the vector space M_{1n} (or M_{n1}) in disguise — it is a vector space by the previous slide.

The smallest vector space, and an infinite-dimensional one

Example 1 — the zero vector space

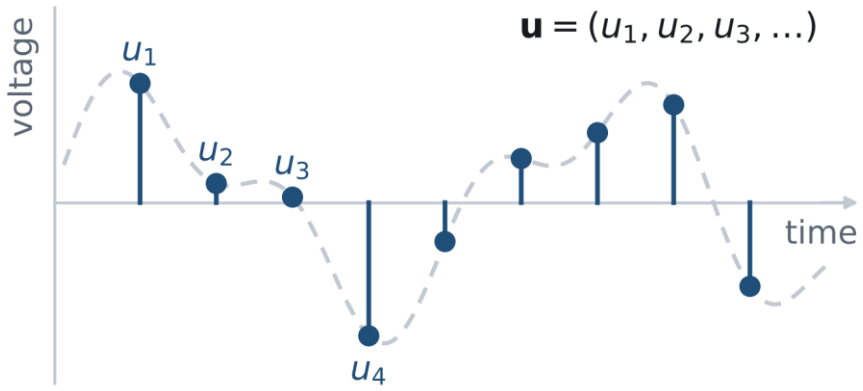
$V = \{\mathbf{0}\}$, a single object, with $\mathbf{0} + \mathbf{0} = \mathbf{0}$ and $k\mathbf{0} = \mathbf{0}$ for every k. All ten axioms hold. A vector space can have just ONE vector; it can never be empty (Axiom 4).

Example 3 — infinite sequences, $\mathbb{R}^\infty$

Vectors $\mathbf{u} = (u_1, u_2, u_3, \dots)$, added and scaled term by term:

$$\mathbf{u} + \mathbf{v} = (u_1 + v_1,\ u_2 + v_2,\ \dots), \quad k\mathbf{u} = (ku_1,\ ku_2,\ \dots)$$

The zero vector is $(0,0,0,\dots)$ and $-\mathbf{u} = (-u_1, -u_2, \dots)$.



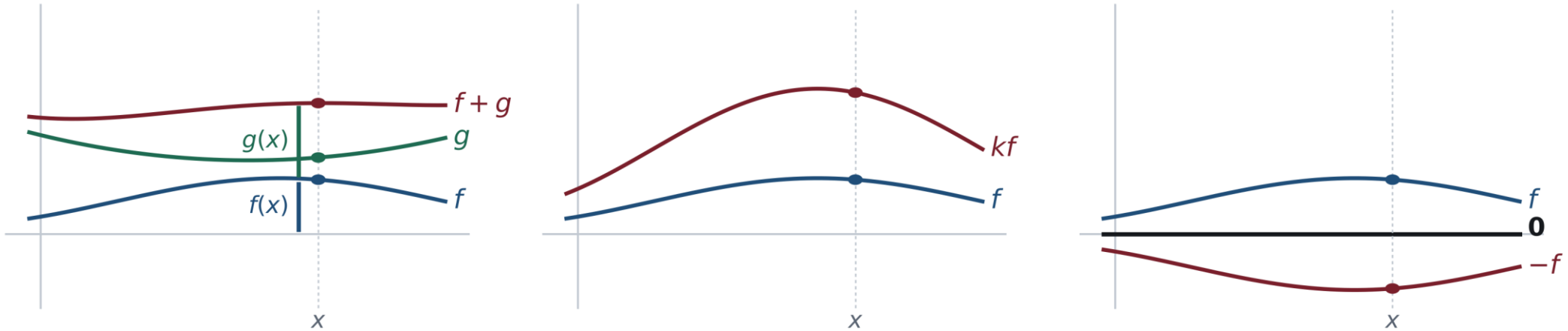
A signal sampled at equal time steps is ONE vector of $\mathbb{R}^\infty$.

EXAMPLE 6

Functions are vectors: $F(-\infty, \infty)$

$V = F(-\infty, \infty)$ is the set of all real-valued functions defined for every real number x. For $f, g \in V$ and a scalar k, define

$$(f + g)(x) = f(x) + g(x), \qquad (kf)(x) = k\,f(x)$$



(a) sum: $(f+g)(x) = f(x) + g(x)$ (b) scalar multiple: $(kf)(x) = k\,f(x)$ (c) negative: $(-f)(x) = -f(x)$

- Think of $f(x)$ as the 'component of f at x': functions are added and scaled component by component, just as in $\mathbb{R}^n$.
- The zero vector is the zero FUNCTION, $\mathbf{0}(x) = 0$ for every x: its graph is the x-axis. The negative of f is $-f$, the reflection of its graph in the x-axis.

One failing axiom is enough

With the usual operations, none of these sets is a vector space:

	Axiom that fails	A specific counterexample
The integers $\mathbb{Z}$	6 — closure under scalars	$1/2 \cdot 1 = 1/2$ is not an integer
n-tuples with positive entries	6 — closure under scalars	$(-1)(1,1) = (-1,-1)$
Polynomials of degree exactly 2	1 — closure under addition	$x^2 + \left(1 + x - x^2\right) = 1 + x$

> Test closure first: an operation that leads OUT of the set is the most common way a set fails to be a vector space — and then nothing else needs checking.

Polynomials of degree exactly 2 also fail closure under scalars: $0 \cdot x^2 = 0$ does not have degree 2.

EXAMPLE 7

Nine axioms hold — and it is still not a vector space

$V = \mathbb{R}^2$ with the STANDARD addition, but with scalar multiplication $k\mathbf{u} = (ku_1,\ 0)$ for $\mathbf{u} = (u_1, u_2)$.

For instance, with $\mathbf{u} = (2,4)$, $\mathbf{v} = (-3,5)$ and $k = 7$: $\mathbf{u} + \mathbf{v} = (-1,9)$ and $7\mathbf{u} = (14,0)$.

	Computation	Result
Axiom 6	$k\mathbf{u} = (ku_1, 0)$ is in $\mathbb{R}^2$	holds
Axiom 7	$k(\mathbf{u} + \mathbf{v}) = (ku_1 + kv_1,\ 0) = k\mathbf{u} + k\mathbf{v}$	holds
Axiom 8	$(k + m)\mathbf{u} = (ku_1 + mu_1,\ 0) = k\mathbf{u} + m\mathbf{u}$	holds
Axiom 9	$k(m\mathbf{u}) = k(mu_1, 0) = (kmu_1, 0) = (km)\mathbf{u}$	holds
Axiom 10	$1\mathbf{u} = (u_1, 0)$, not $\mathbf{u}$ when $u_2 \neq 0$	FAILS

> Axioms 1–5 hold because the addition is standard; Axioms 6–9 hold as shown. Only Axiom 10 fails — for example $1(1,1) = (1,0) \neq (1,1)$ — and one failure is enough: V is NOT a vector space.

EXAMPLE 8

An unusual vector space

$V = \{u \in \mathbb{R} : u > 0\}$, the POSITIVE real numbers. Each vector $\mathbf{u} \in V$ is a positive number u. To keep the two kinds of operation apart, write the vector operations as $\oplus$ and $\odot$:

$$\mathbf{u} \oplus \mathbf{v} = uv \qquad\qquad k \odot \mathbf{u} = u^k$$

Vector addition is ordinary MULTIPLICATION; scalar multiplication is ordinary EXPONENTIATION. For example $2 \oplus 3 = 6$, $1 \oplus 1 = 1$, $2 \odot 3 = 9$, $(-1) \odot 4 = 1/4$.

- Axioms 1, 6: $uv > 0$ and $u^k > 0$ whenever $u, v > 0$ — closure.
- Axiom 4: the zero vector is the NUMBER 1, since $\mathbf{u} \oplus 1 = u \cdot 1 = u$.
- Axiom 5: the negative of $\mathbf{u}$ is $1/u$, since $\mathbf{u} \oplus 1/u = u \cdot 1/u = 1 = \mathbf{0}$.
- Axiom 7: $k \odot (\mathbf{u} \oplus \mathbf{v}) = (uv)^k = u^k v^k = (k \odot \mathbf{u}) \oplus (k \odot \mathbf{v})$.
- Axioms 2, 3, 8, 9, 10 are the laws of multiplication and of exponents, e.g. $(k + m) \odot \mathbf{u} = u^{k+m} = u^k u^m$.

THEOREM 4.1.1

First consequences of the axioms

THEOREM 4.1.1 Let V be a vector space, $\mathbf{u}$ a vector in V and k a scalar. Then (a) $0\mathbf{u} = \mathbf{0}$; (b) $k\mathbf{0} = \mathbf{0}$; (c) $(-1)\mathbf{u} = -\mathbf{u}$; (d) if $k\mathbf{u} = \mathbf{0}$, then $k = 0$ or $\mathbf{u} = \mathbf{0}$.

- Obvious in $\mathbb{R}^n$ — but in a general vector space none of these is an axiom, so each must be PROVED from the axioms alone. Watch the two zeros in (a): the NUMBER 0 on the left, the zero VECTOR $\mathbf{0}$ on the right.

Proof of (a) — the number over each $=$ is the axiom used

$$0\mathbf{u} = (0+0)\mathbf{u} \overset{(8)}{=} 0\mathbf{u} + 0\mathbf{u}$$

Add $-0\mathbf{u}$, the negative of $0\mathbf{u}$ (Axiom 5), and use the line above in the second step:

$$\mathbf{0} \overset{(5)}{=} 0\mathbf{u} + (-0\mathbf{u}) = (0\mathbf{u} + 0\mathbf{u}) + (-0\mathbf{u}) \overset{(3)}{=} 0\mathbf{u} + \big(0\mathbf{u} + (-0\mathbf{u})\big) \overset{(5)}{=} 0\mathbf{u} + \mathbf{0} \overset{(4)}{=} 0\mathbf{u}$$

PROOF

Parts (c), (b) and (d)

(c) $(-1)\mathbf{u} = -\mathbf{u}$

$$\mathbf{u} + (-1)\mathbf{u} \overset{(10)}{=} 1\mathbf{u} + (-1)\mathbf{u} \overset{(8)}{=} \big(1 + (-1)\big)\mathbf{u} = 0\mathbf{u} \overset{(a)}{=} \mathbf{0}$$

So $(-1)\mathbf{u}$ is a negative of $\mathbf{u}$ — and a vector has only one negative: if $\mathbf{u} + \mathbf{w} = \mathbf{0}$, then $\mathbf{w} = (-\mathbf{u} + \mathbf{u}) + \mathbf{w} = -\mathbf{u} + (\mathbf{u} + \mathbf{w}) = -\mathbf{u} + \mathbf{0} = -\mathbf{u}$.

(b) $k\mathbf{0} = \mathbf{0}$

$$k\mathbf{0} \overset{(4)}{=} k(\mathbf{0} + \mathbf{0}) \overset{(7)}{=} k\mathbf{0} + k\mathbf{0}$$

Now add $-k\mathbf{0}$ to both sides, exactly as in (a).

(d) **If** $k\mathbf{u} = \mathbf{0}$**, then** $k = 0$ **or** $\mathbf{u} = \mathbf{0}$

$$k \neq 0: \qquad \mathbf{u} \overset{(10)}{=} 1\mathbf{u} = \left(\frac{1}{k}\, k\right)\mathbf{u} \overset{(9)}{=} \frac{1}{k}(k\mathbf{u}) = \frac{1}{k}\,\mathbf{0} \overset{(b)}{=} \mathbf{0}$$

SUMMARY

Vector spaces met so far

	The vectors	Zero vector	Negative of $\mathbf{u}$
$\mathbb{R}^n$	n-tuples $(u_1, \dots, u_n)$	$(0, \dots, 0)$	$(-u_1, \dots, -u_n)$
M_{mn}	$m \times n$ matrices A	the zero matrix O	$-A$
$\mathbb{R}^\infty$	infinite sequences	$(0,0,0, \dots)$	$(-u_1, -u_2, \dots)$
$F(-\infty, \infty)$	functions f on $\mathbb{R}$	$\mathbf{0}(x) = 0$ for all x	$(-f)(x) = -f(x)$
$\{\mathbf{0}\}$	a single object	$\mathbf{0}$	$\mathbf{0}$
$(0, \infty)$ **with** $\oplus, \odot$	positive numbers u	the number 1	$1/u$

Coming in Section 4.2: P_n, the polynomials of degree at most n — and many more spaces found INSIDE these.

SECTION 4.2

Subspaces

GOAL Recognise vector spaces sitting inside larger ones, and test them with three simple conditions. Then meet the two great sources of subspaces: spans of vectors, and solution sets of homogeneous systems.

1. Subspaces, and the subspace test
2. Subspaces of $\mathbb{R}^2$, $\mathbb{R}^3$ and M_{nn} — and sets that are not
3. Subspaces of functions; intersections
4. Linear combinations and spans
5. Is $\mathbf{w}$ a linear combination? Do the vectors span $\mathbb{R}^n$?
6. Solution spaces of homogeneous systems

DEFINITION 1

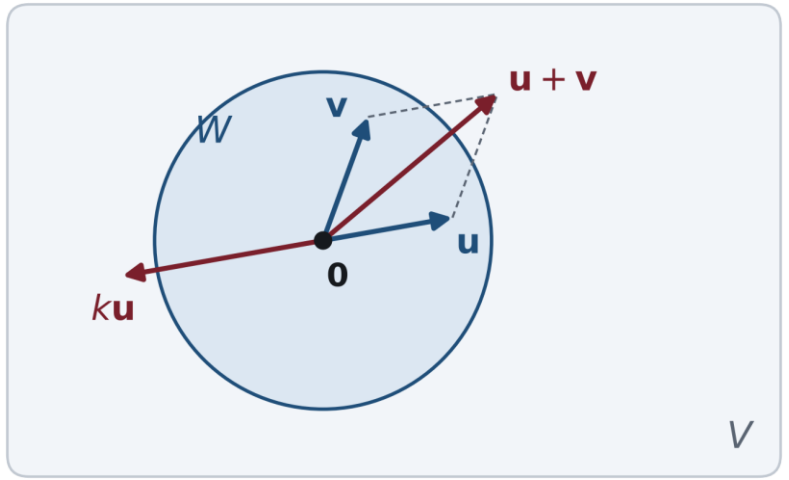
Subspaces

A subset W of a vector space V is a SUBSPACE of V if W is itself a vector space under the addition and scalar multiplication of V.

Most axioms are INHERITED from V: $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ holds for all vectors of V, so in particular for those in W. The same goes for Axioms 3, 7, 8, 9 and 10.

NOT inherited — these must be checked in W:

- Axioms 1 and 6: W is closed under addition and under scalar multiplication;
- Axioms 4 and 5: $\mathbf{0} \in W$, and $-\mathbf{u} \in W$ for every $\mathbf{u} \in W$.



$\mathbf{u}, \mathbf{v} \in W$, but $\mathbf{u} + \mathbf{v} \notin W$ and $k\mathbf{u} \notin W$: this W is not closed.

THEOREM 4.2.1

When is a subset a subspace?

THEOREM 4.2.1 A subset W of a vector space V is a subspace of V if and only if

1) $\mathbf{0} \in W$ (the zero vector of V);

2) if $\mathbf{u}, \mathbf{v} \in W$, then $\mathbf{u} + \mathbf{v} \in W$;

3) if $\mathbf{u} \in W$ and k is a scalar, then $k\mathbf{u} \in W$.

Proof

($\Rightarrow$) A subspace is a vector space, so Axioms 1 and 6 hold in W: these are 2) and 3). By Axiom 4, W is not empty, so take any $\mathbf{u} \in W$. By 3), $0\mathbf{u} \in W$, and $0\mathbf{u} = \mathbf{0}$ by Theorem 4.1.1(a). So 1) holds.

($\Leftarrow$) Condition 1) gives Axiom 4, and 2), 3) are Axioms 1 and 6. Axioms 2, 3 and 7–10 are inherited from V. That leaves Axiom 5: if $\mathbf{u} \in W$, then by Theorem 4.1.1(c) and 3),

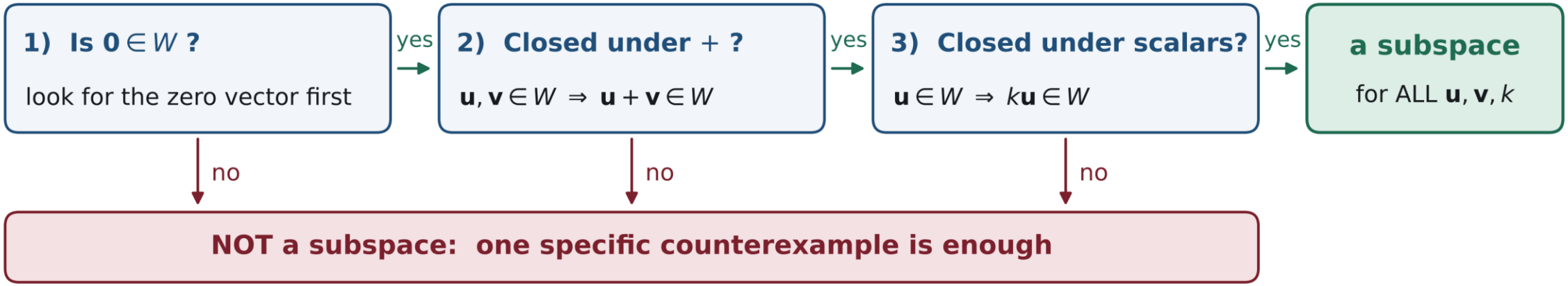
$$-\mathbf{u} = (-1)\mathbf{u} \in W.$$

So all ten axioms hold in W: it is a subspace. ■

In words: W is a subspace exactly when it contains $\mathbf{0}$ and is CLOSED under addition and under scalar multiplication.

IN PRACTICE

The subspace test, step by step

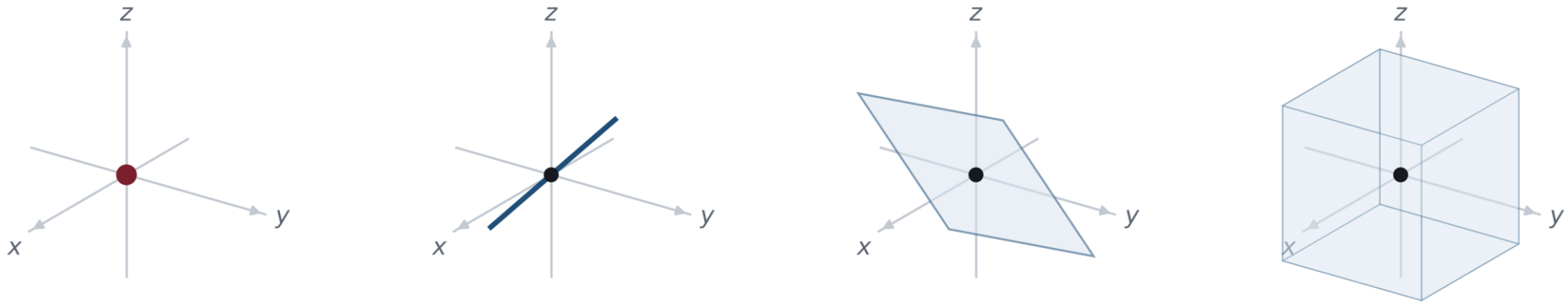


- Check 1) first. If $\mathbf{0} \notin W$, stop: W is NOT a subspace, and nothing else needs checking.
- To PROVE 2) and 3), take GENERAL vectors $\mathbf{u}, \mathbf{v} \in W$ and a general scalar k. Checking one or two examples proves nothing.
- To DISPROVE 2) or 3), one specific pair $\mathbf{u}, \mathbf{v} \in W$, or one $\mathbf{u} \in W$ and one scalar k, is enough.

WARNING $\mathbf{0} \in W$ is necessary, but not sufficient: the first quadrant $\{(x, y) \in \mathbb{R}^2 : x \geq 0,\ y \geq 0\}$ contains $\mathbf{0}$, but it is not a subspace (Example 4).

The subspaces of $\mathbb{R}^2$ and $\mathbb{R}^3$

- Every vector space V has at least two subspaces: V itself and the ZERO SUBSPACE $\{\mathbf{0}\}$, since $\mathbf{0} + \mathbf{0} = \mathbf{0}$ and $k\mathbf{0} = \mathbf{0}$ (Example 1).
- Adding or scaling vectors that lie on a line through the origin, or in a plane through the origin, never leaves it: lines through $\mathbf{0}$ and planes through $\mathbf{0}$ are subspaces (Examples 2 and 3).



$\{\mathbf{0}\}$ | a line through $\mathbf{0}$ | a plane through $\mathbf{0}$ | all of $\mathbb{R}^3$

TABLE 1 Subspaces of $\mathbb{R}^2$: $\{\mathbf{0}\}$, lines through $\mathbf{0}$, $\mathbb{R}^2$. Subspaces of $\mathbb{R}^3$: $\{\mathbf{0}\}$, lines through $\mathbf{0}$, planes through $\mathbf{0}$, $\mathbb{R}^3$. Section 4.5 shows there are no others.

EXAMPLE

A line through the origin — and one that misses it

Which of these subsets of $\mathbb{R}^2$ is a subspace?

(a) $W = \{(x, y) \in \mathbb{R}^2 : x + 2y = 0\}$

Solving $x + 2y = 0$ gives $W = \{(-2t,\ t) : t \in \mathbb{R}\}$, a line through the origin. It contains $(0,0)$ (take $t = 0$), and

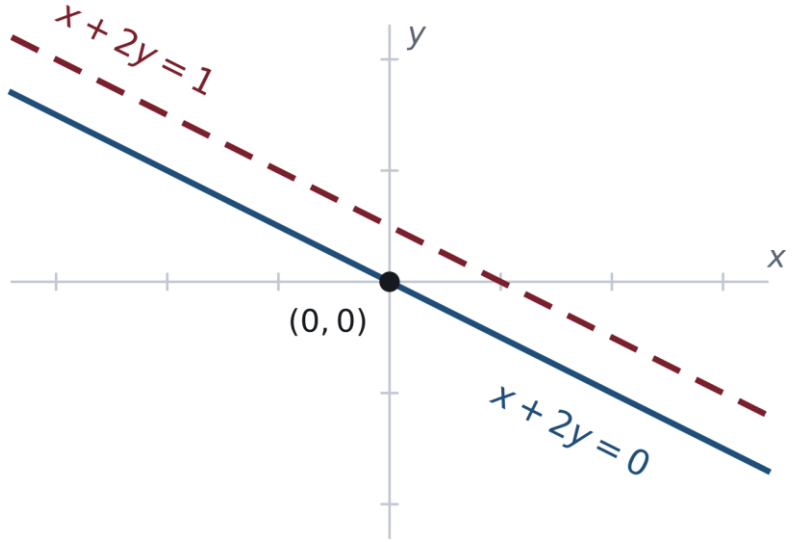
$$(-2t_1, t_1) + (-2t_2, t_2) = (-2(t_1 + t_2),\ t_1 + t_2)$$

$$k\,(-2t,\ t) = (-2(kt),\ kt)$$

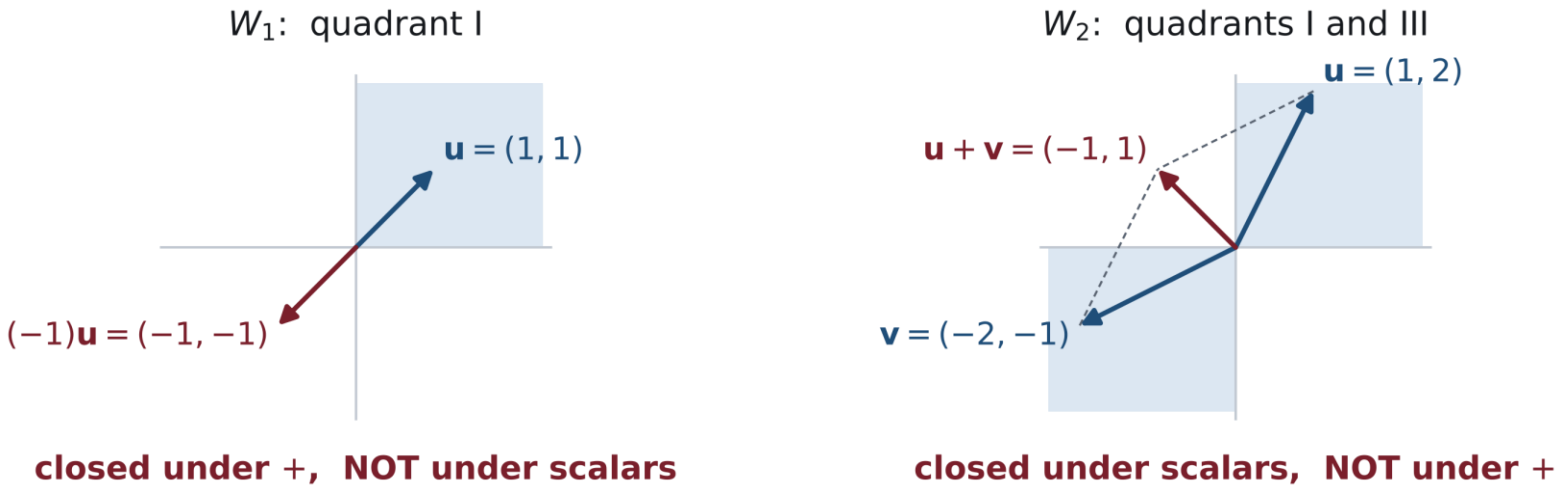
are again in W. A SUBSPACE.

(b) $W = \{(x, y) \in \mathbb{R}^2 : x + 2y = 1\}$

$0 + 2 \cdot 0 = 0 \neq 1$, so $(0,0) \notin W$. NOT a subspace: there is nothing more to check.



The two closure conditions are independent

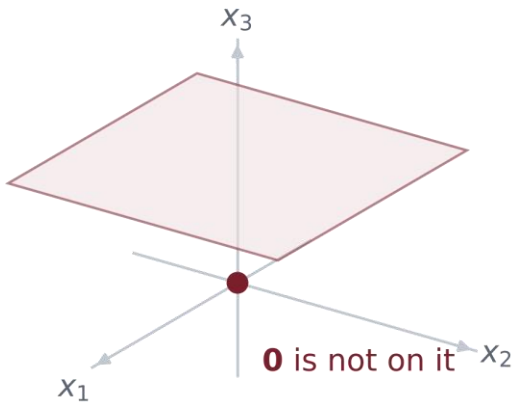


- Quadrant I, $W_1 = \{(x, y) \in \mathbb{R}^2 : x \geq 0,\ y \geq 0\}$, is closed under addition: sums of nonnegative numbers are nonnegative. But $(1,1) \in W_1$ and $(-1)(1,1) = (-1,-1) \notin W_1$ (Anton's Example 4).
- Quadrants I and III, $W_2 = \{(x, y) \in \mathbb{R}^2 : xy \geq 0\}$, are closed under scalars, since $(kx)(ky) = k^2xy \geq 0$. But $(1,2), (-2,-1) \in W_2$ and $(1,2) + (-2,-1) = (-1,1) \notin W_2$.

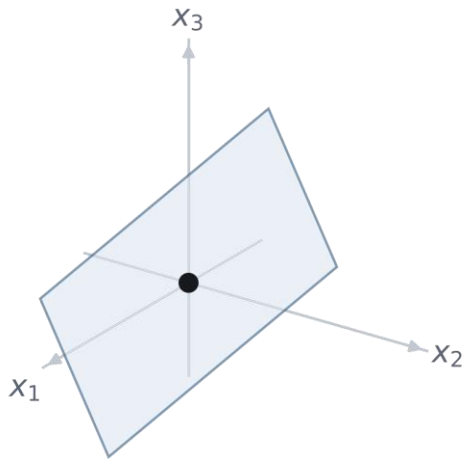
> Both sets contain $\mathbf{0}$, and each one satisfies one of the conditions 2), 3), yet neither is a subspace. Check ALL three conditions, every time.

EXAMPLE

Two planes in $\mathbb{R}^3$



(a) the plane $x_3 = 1$



(b) the plane $x_2 = x_1 + x_3$

(a) $W = \{(x_1, x_2, 1): x_1, x_2 \in \mathbb{R}\}$

Every vector of W has third entry 1, so $(0,0,0) \notin W$. NOT a subspace: the plane $x_3 = 1$ misses the origin.

(b) $W = \{(x_1, x_1 + x_3, x_3): x_1, x_3 \in \mathbb{R}\}$

Take $x_1 = x_3 = 0$: $\mathbf{0} \in W$. If $\mathbf{u} = (u_1, u_1 + u_3, u_3)$ and $\mathbf{v} = (v_1, v_1 + v_3, v_3)$ are in W, the middle entry of $\mathbf{u} + \mathbf{v}$ is $(u_1 + v_1) + (u_3 + v_3)$ and that of $k\mathbf{u}$ is $ku_1 + ku_3$. Both keep the form 'first + last', so $\mathbf{u} + \mathbf{v} \in W$ and $k\mathbf{u} \in W$. A SUBSPACE: the plane $x_2 = x_1 + x_3$.

EXAMPLE 5

Subspaces of M_{nn}

The SYMMETRIC matrices, $W = \{A \in M_{nn}: A^T = A\}$. For $A_1, A_2 \in W$ and a scalar k:

$$O^T = O, \qquad (A_1 + A_2)^T = A_1^T + A_2^T = A_1 + A_2, \qquad (kA_1)^T = kA_1^T = kA_1$$

So $O \in W$, $A_1 + A_2 \in W$ and $kA_1 \in W$ (Theorem 1.7.2): W is a subspace of M_{nn}.

- In the same way the UPPER TRIANGULAR, the LOWER TRIANGULAR and the DIAGONAL $n \times n$ matrices are subspaces of M_{nn}, because sums and scalar multiples keep the zeros where they are:

$$\begin{bmatrix} a & b \\ 0 & d \end{bmatrix} + \begin{bmatrix} a' & b' \\ 0 & d' \end{bmatrix} = \begin{bmatrix} a + a' & b + b' \\ 0 & d + d' \end{bmatrix}, \qquad k\begin{bmatrix} a & b \\ 0 & d \end{bmatrix} = \begin{bmatrix} ka & kb \\ 0 & kd \end{bmatrix}$$

- So is $\{A \in M_{nn}: \mathrm{tr}(A) = 0\}$, since $\mathrm{tr}(A + B) = \mathrm{tr}(A) + \mathrm{tr}(B)$ and $\mathrm{tr}(kA) = k\mathrm{tr}(A)$ (Exercise 2(c)).

EXAMPLE 6

Invertible and singular matrices: NOT subspaces

The invertible matrices

$U = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}$ and $V = \begin{bmatrix} -1 & 2 \\ -2 & 5 \end{bmatrix}$ are invertible, but

$$0U = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \qquad U + V = \begin{bmatrix} 0 & 4 \\ 0 & 10 \end{bmatrix}$$

are not: the zero matrix is not invertible, and $U + V$ has a column of zeros. Closure fails BOTH ways — and Step 1 already fails.

The singular matrices

The zero matrix is singular, and so is any multiple of a singular matrix ($\det(kA) = k^n \det(A) = 0$). But

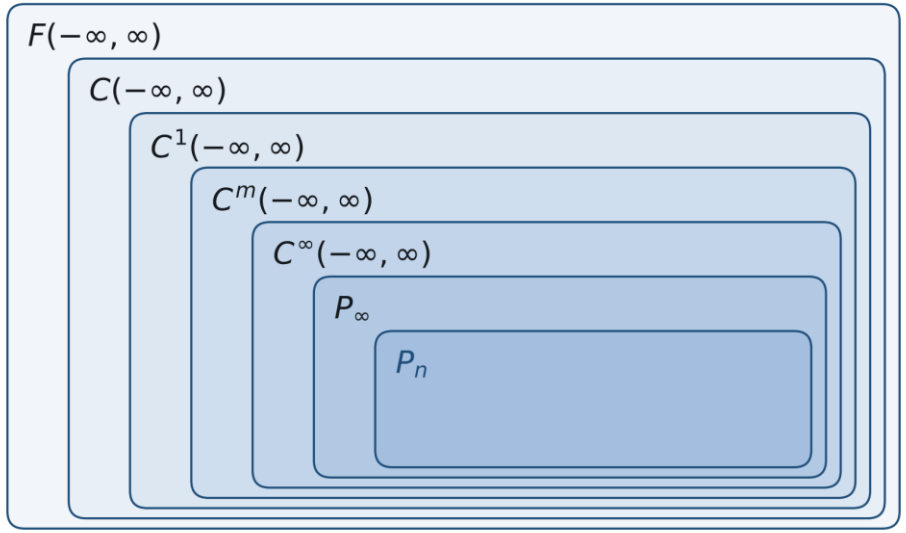
$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

adds two singular matrices and gets I, which is invertible: NOT closed under addition.

Subspaces of functions

- $C(-\infty, \infty)$: the CONTINUOUS functions. Sums and constant multiples of continuous functions are continuous.
- C^1, C^m, C^∞: functions whose first, first m, or all derivatives exist and are continuous.
- P_∞: all polynomials. P_n: the polynomials of degree at most n, $P_n = \{a_0 + a_1x + \cdots + a_nx^n : a_0, a_1, \ldots, a_n \in \mathbb{R}\}$.

The polynomials of degree EXACTLY n do not form a subspace. For $n = 2$: $(1 + 2x + 3x^2) + (5 + 7x - 3x^2) = 6 + 9x$ has degree 1.

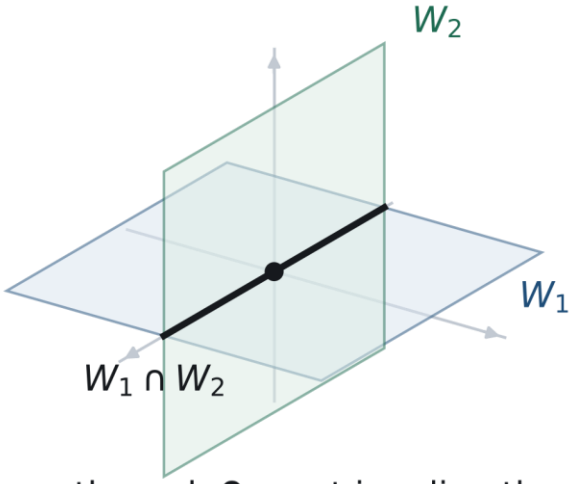


Each space is a subspace of every space that contains it.

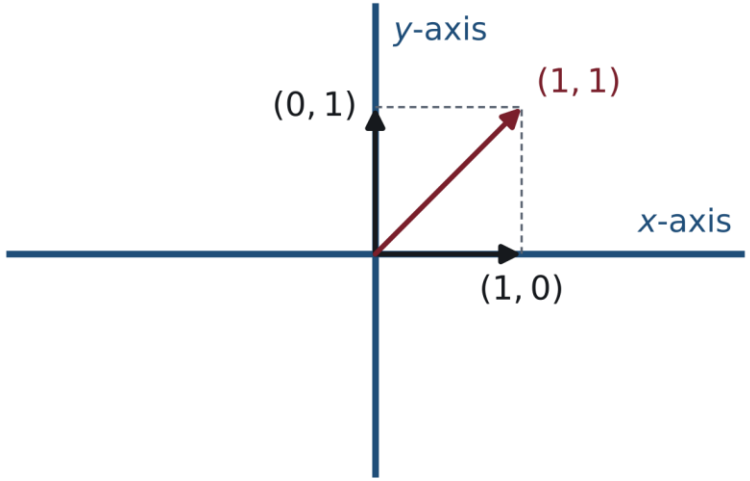
Intersections of subspaces are subspaces

THEOREM 4.2.2 If $W_1, W_2, \ldots, W_r$ are subspaces of a vector space V, then their intersection $W_1 \cap W_2 \cap \cdots \cap W_r$ is also a subspace of V.

Proof. Let $W = W_1 \cap \cdots \cap W_r$. Each W_i contains $\mathbf{0}$, so $\mathbf{0} \in W$. If $\mathbf{u}, \mathbf{v} \in W$ and k is a scalar, then $\mathbf{u}, \mathbf{v} \in W_i$ for every i. Each W_i is a subspace, so $\mathbf{u} + \mathbf{v} \in W_i$ and $k\mathbf{u} \in W_i$ for every i; that is, $\mathbf{u} + \mathbf{v} \in W$ and $k\mathbf{u} \in W$. By Theorem 4.2.1, W is a subspace. ■



two planes through **0** meet in a line through **0**



union of the two axes: (1, 1) lies on neither

The UNION of two subspaces is usually NOT a subspace (right?).

Linear combinations and span

DEFINITION 2 $\mathbf{w}$ is a LINEAR COMBINATION of $\mathbf{v}_1, \dots, \mathbf{v}_r$ if $\mathbf{w} = k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \cdots + k_r\mathbf{v}_r$ for some scalars $k_1, \dots, k_r$, the COEFFICIENTS.

THEOREM 4.2.3 Let $S = \{\mathbf{w}_1, \dots, \mathbf{w}_r\}$ be a nonempty set of vectors in V. (a) The set W of ALL linear combinations of $\mathbf{w}_1, \dots, \mathbf{w}_r$ is a subspace of V. (b) It is the SMALLEST subspace of V containing S: every subspace that contains S contains W.

DEFINITION 3 This W is the subspace SPANNED by S, written $\text{span}(S)$ or $\text{span}\{\mathbf{w}_1, \dots, \mathbf{w}_r\}$. So $\text{span}\{\mathbf{w}_1, \dots, \mathbf{w}_r\} = \{k_1\mathbf{w}_1 + \cdots + k_r\mathbf{w}_r : k_1, \dots, k_r \in \mathbb{R}\}$, and we say that $\mathbf{w}_1, \dots, \mathbf{w}_r$ SPAN W.

Taking every $k_i = 0$ shows that $\mathbf{0} \in \text{span}(S)$ for every S.

PROOF

Why a span is the smallest subspace containing S

(a) W is a subspace

$\mathbf{0} = 0\mathbf{w}_1 + \cdots + 0\mathbf{w}_r \in W$. Take $\mathbf{u} = c_1\mathbf{w}_1 + \cdots + c_r\mathbf{w}_r$ and $\mathbf{v} = d_1\mathbf{w}_1 + \cdots + d_r\mathbf{w}_r$ in W, and a scalar k. Then

$$\mathbf{u} + \mathbf{v} = (c_1 + d_1)\mathbf{w}_1 + \cdots + (c_r + d_r)\mathbf{w}_r, \qquad k\mathbf{u} = (kc_1)\mathbf{w}_1 + \cdots + (kc_r)\mathbf{w}_r$$

are again linear combinations of $\mathbf{w}_1, \ldots, \mathbf{w}_r$, so $\mathbf{u} + \mathbf{v} \in W$ and $k\mathbf{u} \in W$. By Theorem 4.2.1, W is a subspace.

(b) W is the smallest

Let U be any subspace of V that contains $\mathbf{w}_1, \ldots, \mathbf{w}_r$. Since U is closed under scalar multiplication, $k_i\mathbf{w}_i \in U$ for each i; since U is closed under addition, $k_1\mathbf{w}_1 + \cdots + k_r\mathbf{w}_r \in U$. So every vector of W lies in U: $W \subseteq U$. ■

Consequence: to show that a set is a subspace, it is enough to write it as a span.

Standard spanning sets

$\mathbb{R}^n$: the standard unit vectors

$$(v_1, v_2, \dots, v_n) = v_1\mathbf{e}_1 + v_2\mathbf{e}_2 + \cdots + v_n\mathbf{e}_n, \qquad (a, b, c) = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$$

with $\mathbf{e}_1 = (1,0,\dots,0)$, …, $\mathbf{e}_n = (0,\dots,0,1)$, and $\mathbf{i}, \mathbf{j}, \mathbf{k}$ in $\mathbb{R}^3$. So $\mathbb{R}^n = \text{span}\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$.

P_n: the powers of x

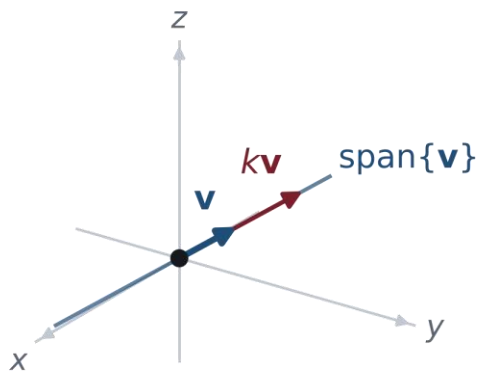
$$p(x) = a_0 \cdot 1 + a_1 x + \cdots + a_n x^n, \qquad P_n = \text{span}\{1, x, x^2, \dots, x^n\}$$

M_{22}: the four matrix units

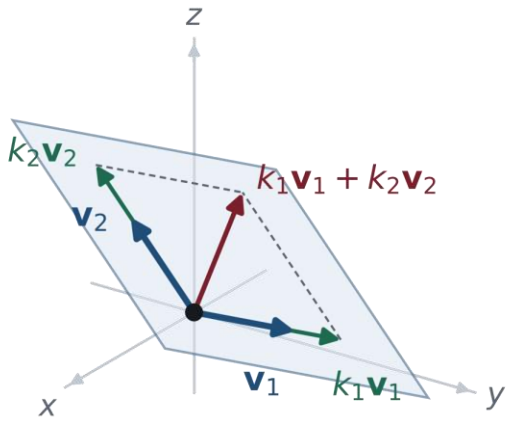
$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + d\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

EXAMPLE 12

A geometric view of spanning



(a) span{**v**}: the line through **0** along **v**



(b) span{$\mathbf{v}_1, \mathbf{v}_2$}: the plane through **0**

(a) $\mathbf{v} \neq \mathbf{0}$: $\text{span}\{\mathbf{v}\} = \{k\mathbf{v}: k \in \mathbb{R}\}$, the line through the origin along $\mathbf{v}$.

(b) $\mathbf{v}_1, \mathbf{v}_2$ nonzero and NOT parallel: $\text{span}\{\mathbf{v}_1, \mathbf{v}_2\} = \{k_1\mathbf{v}_1 + k_2\mathbf{v}_2: k_1, k_2 \in \mathbb{R}\}$, the plane through the origin that contains them.

> Spanning sets are not unique: ANY nonzero vector on the line spans it, and ANY two non-parallel vectors in the plane span it.

TECHNIQUE

Write the set as a span — then it is a subspace

Write a general element, pull the free parameters out as coefficients, and read off a spanning set.

$$W_1 = \{(x_1,\ x_1 + x_3,\ x_3) : x_1, x_3 \in \mathbb{R}\}$$

$$(x_1,\ x_1 + x_3,\ x_3) = x_1(1,1,0) + x_3(0,1,1), \qquad \text{so} \qquad W_1 = \text{span}\{(1,1,0),\ (0,1,1)\}$$

$$W_2 = \{(a - 3b,\ b - a,\ a,\ b) : a, b \in \mathbb{R}\}$$

$$(a - 3b,\ b - a,\ a,\ b) = a(1,-1,1,0) + b(-3,1,0,1), \qquad \text{so} \qquad W_2 = \text{span}\{(1,-1,1,0),\ (-3,1,0,1)\}$$

$W_3 = \{A \in M_{22} : A^T = A\}$**, the symmetric** 2×2 **matrices**

$$\begin{bmatrix} a & b \\ b & d \end{bmatrix} = a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + d\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \qquad \text{so} \qquad W_3 = \text{span}\left\{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right\}$$

All three are subspaces by Theorem 4.2.3, with no axiom checked. (W_1 is the plane $x_2 = x_1 + x_3$ of the earlier example.)

This works when every entry is a combination of the parameters with NO constant term. $(x_1, x_2, 1)$ cannot be written this way, and $\{(x_1, x_2, 1) : x_1, x_2 \in \mathbb{R}\}$ is not a subspace.

EXAMPLE 14

Is w a linear combination of u and v?

Let $\mathbf{u} = (1,2,-1)$ and $\mathbf{v} = (6,4,2)$. Is $\mathbf{w} = (9,2,7)$ a linear combination of $\mathbf{u}$ and $\mathbf{v}$? We look for scalars k_1, k_2 with $\mathbf{w} = k_1\mathbf{u} + k_2\mathbf{v}$.

Step 1. Write out the vector equation, and simplify the right side

$$(9,2,7) = k_1(1,2,-1) + k_2(6,4,2) = (k_1 + 6k_2,\ 2k_1 + 4k_2,\ -k_1 + 2k_2)$$

Step 2. Equate components: one equation for each component

$$\begin{aligned} k_1 + 6k_2 &= 9 \\ 2k_1 + 4k_2 &= 2 \\ -k_1 + 2k_2 &= 7 \end{aligned}$$

Step 3. Row reduce the augmented matrix $[\,\mathbf{u}\ \ \mathbf{v} \mid \mathbf{w}\,]$: its columns are u, v and w

$$\left[\begin{array}{cc|c} 1 & 6 & 9 \\ 2 & 4 & 2 \\ -1 & 2 & 7 \end{array}\right] \xrightarrow{A_{1,2}^{(-2)},\ A_{1,3}^{(1)}} \left[\begin{array}{cc|c} 1 & 6 & 9 \\ 0 & -8 & -16 \\ 0 & 8 & 16 \end{array}\right] \xrightarrow{A_{2,3}^{(1)}} \left[\begin{array}{cc|c} 1 & 6 & 9 \\ 0 & -8 & -16 \\ 0 & 0 & 0 \end{array}\right]$$

Consistent: $k_2 = 2$, and then $k_1 = 9 - 6(2) = -3$. So $\mathbf{w} = -3\mathbf{u} + 2\mathbf{v}$. Check: $-3(1,2,-1) + 2(6,4,2) = (9,2,7)$.

Which vectors ARE linear combinations of u and v?

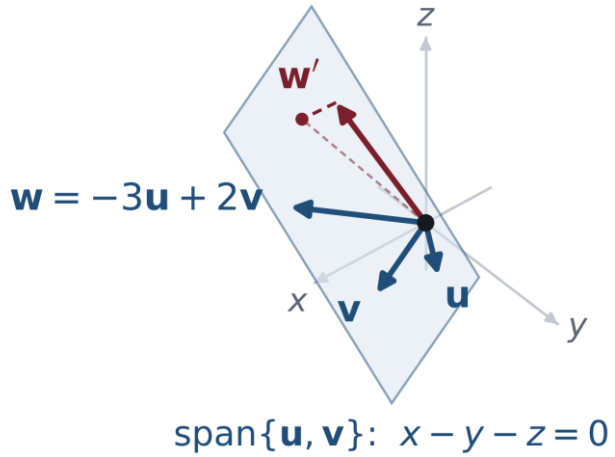
Now try $\mathbf{w}' = (4,-1,8)$, or better, ANY vector $\mathbf{b} = (b_1, b_2, b_3)$. The columns $\mathbf{u}, \mathbf{v}$ do not change, so the same row operations apply:

$$\left[\begin{array}{cc|c} 1 & 6 & b_1 \\ 2 & 4 & b_2 \\ -1 & 2 & b_3 \end{array}\right] \overset{A_{1,2}^{(-2)},\, A_{1,3}^{(1)}}{\rightarrow} \left[\begin{array}{cc|c} 1 & 6 & b_1 \\ 0 & -8 & -2b_1 + b_2 \\ 0 & 8 & b_1 + b_3 \end{array}\right] \overset{A_{2,3}^{(1)}}{\rightarrow} \left[\begin{array}{cc|c} 1 & 6 & b_1 \\ 0 & -8 & -2b_1 + b_2 \\ 0 & 0 & -b_1 + b_2 + b_3 \end{array}\right]$$

The last row is the equation $0 = -b_1 + b_2 + b_3$. So $\mathbf{b}$ is a linear combination of $\mathbf{u}$ and $\mathbf{v}$ exactly when $b_1 - b_2 - b_3 = 0$.

- $\mathbf{w} = (9,2,7)$: $9 - 2 - 7 = 0$. A linear combination, as we found.
- $\mathbf{w}' = (4,-1,8)$: the last row becomes $[\,0 \;\; 0 \mid 3\,]$, the equation $0 = 3$. No solution, so $\mathbf{w}'$ is NOT a linear combination of $\mathbf{u}$ and $\mathbf{v}$.

> In pictures: $\text{span}\{\mathbf{u}, \mathbf{v}\}$ is the plane $x - y - z = 0$ through the origin (Example 12). $\mathbf{w}$ lies in it and $\mathbf{w}'$ does not.



EXAMPLE

The coefficients need not be unique

$\mathbf{v}_1 = (1,2,3)$, $\mathbf{v}_2 = (0,1,2)$, $\mathbf{v}_3 = (-1,0,1)$ and $\mathbf{w} = (1,1,1)$. As in Example 14, the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are the columns and $\mathbf{w}$ is the right side:

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 2 & 1 & 0 & 1 \\ 3 & 2 & 1 & 1 \end{array}\right] \xrightarrow{A_{1,2}^{(-2)},\, A_{1,3}^{(-3)}} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 0 & 1 & 2 & -1 \\ 0 & 2 & 4 & -2 \end{array}\right] \xrightarrow{A_{2,3}^{(-2)}} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$k_3 = t$ is free: $k_1 = 1 + t$, $k_2 = -1 - 2t$. EVERY t gives a way to write $\mathbf{w}$:

$$t = 0\text{:}\ \ \mathbf{w} = \mathbf{v}_1 - \mathbf{v}_2, \qquad\qquad t = 1\text{:}\ \ \mathbf{w} = 2\mathbf{v}_1 - 3\mathbf{v}_2 + \mathbf{v}_3$$

But $(1,-2,2)$ is no combination at all: the same row operations end with the row $[\,0\ \ 0\ \ 0 \mid 7\,]$.

> Many ways to write $\mathbf{w}$ means the $\mathbf{v}$'s are redundant: here $\mathbf{v}_3 = -\mathbf{v}_1 + 2\mathbf{v}_2$. Section 4.3 (linear independence) is about exactly this.

EXAMPLE 15

Do $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ span $\mathbb{R}^3$?

$\mathbf{v}_1 = (1,1,2)$, $\mathbf{v}_2 = (1,0,1)$, $\mathbf{v}_3 = (2,1,3)$ span $\mathbb{R}^3$ exactly when $k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + k_3\mathbf{v}_3 = \mathbf{b}$ is consistent for EVERY $\mathbf{b} = (b_1, b_2, b_3)$ — a square system with coefficient matrix

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 3 \end{bmatrix}, \qquad \det(A) = 1(0-1) - 1(3-2) + 2(1-0) = 0$$

By Theorem 2.3.8, parts (e) and (g), $A\mathbf{x} = \mathbf{b}$ is consistent for every $\mathbf{b}$ exactly when $\det(A) \neq 0$. Here $\det(A) = 0$: the vectors do NOT span $\mathbb{R}^3$.

> Look closer: $\mathbf{v}_3 = \mathbf{v}_1 + \mathbf{v}_2$. All three vectors lie in the plane $x + y - z = 0$, and a vector off that plane, such as $\mathbf{b} = (1,0,0)$, is never reached.

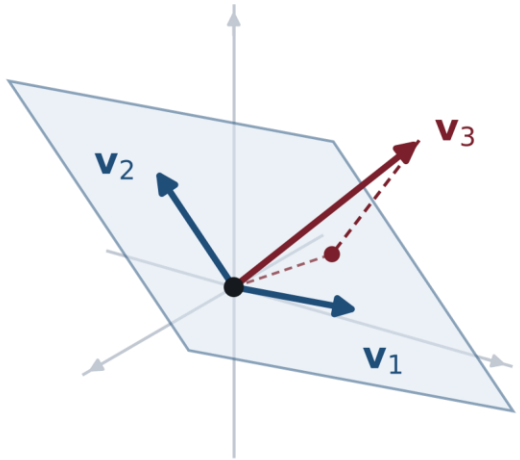
For n vectors in $\mathbb{R}^n$: put them in the COLUMNS of A. They span $\mathbb{R}^n$ exactly when $\det(A) \neq 0$.

CONTRAST

A small change, a big difference

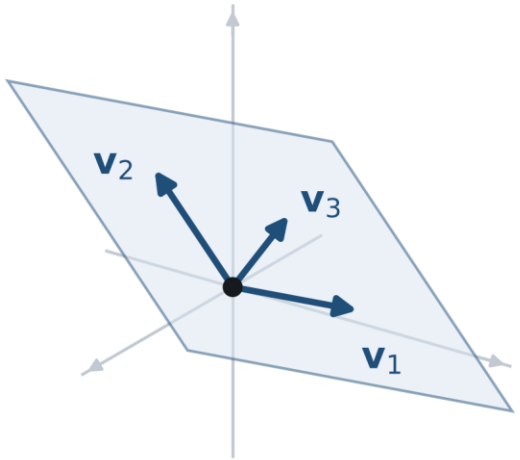
	the set	$\det[\,\mathbf{v}_1\ \mathbf{v}_2\ \mathbf{v}_3\,]$	its span
S_1	$\{(1,2,3),\ (0,1,2),\ (-2,0,1)\}$	-1	all of $\mathbb{R}^3$
S_2	$\{(1,2,3),\ (0,1,2),\ (-1,0,1)\}$	0	a plane only

$\mathbf{v}_3$ is NOT in span$\{\mathbf{v}_1, \mathbf{v}_2\}$



span$\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} = \mathbb{R}^3$

$\mathbf{v}_3$ IS in span$\{\mathbf{v}_1, \mathbf{v}_2\}$



span$\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} =$ the plane

Only the third vector differs. In S_2 it equals $-\mathbf{v}_1 + 2\mathbf{v}_2$ and adds nothing new; in S_1 it points out of the plane of $\mathbf{v}_1, \mathbf{v}_2$. (The picture is schematic.)

THEOREM 4.2.4

Solution spaces of homogeneous systems

> **THEOREM 4.2.4** If A is an $m \times n$ matrix, the solution set $\{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} = \mathbf{0}\}$ of the homogeneous system $A\mathbf{x} = \mathbf{0}$ is a subspace of $\mathbb{R}^n$, called the SOLUTION SPACE of the system.

Proof

$\mathbf{x} = \mathbf{0}$ is a solution. If $A\mathbf{x}_1 = \mathbf{0}$ and $A\mathbf{x}_2 = \mathbf{0}$, then for every scalar k

$$A(\mathbf{x}_1 + \mathbf{x}_2) = A\mathbf{x}_1 + A\mathbf{x}_2 = \mathbf{0} + \mathbf{0} = \mathbf{0}, \qquad A(k\mathbf{x}_1) = k\,A\mathbf{x}_1 = k\mathbf{0} = \mathbf{0}$$

so $\mathbf{x}_1 + \mathbf{x}_2$ and $k\mathbf{x}_1$ are solutions too. ■

For a NONHOMOGENEOUS system ($\mathbf{b} \neq \mathbf{0}$) the solution set is never a subspace: $A\mathbf{0} = \mathbf{0} \neq \mathbf{b}$, so $\mathbf{0}$ is not a solution.

Example: $\{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 + 2x_2 + 3x_3 = 0,\ 4x_1 + 5x_2 + 6x_3 = 0\}$ is a subspace of $\mathbb{R}^3$: it is the solution space of $A\mathbf{x} = \mathbf{0}$ with $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$.

EXAMPLE 16

Every kind of subspace of $\mathbb{R}^3$ is a solution space

$$\text{(a)}\ \begin{bmatrix} 1 & -2 & 3 \\ 2 & -4 & 6 \\ 3 & -6 & 9 \end{bmatrix} \qquad \text{(b)}\ \begin{bmatrix} 1 & -2 & 3 \\ -3 & 7 & -8 \\ -2 & 4 & -6 \end{bmatrix} \qquad \text{(c)}\ \begin{bmatrix} 1 & -2 & 3 \\ -3 & 7 & -8 \\ 4 & 1 & 2 \end{bmatrix} \qquad \text{(d)}\ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

are the coefficient matrices of four homogeneous systems $A\mathbf{x} = \mathbf{0}$ in x, y, z:

	Solutions	Solution space
(a)	$x = 2s - 3t,\ \ y = s,\ \ z = t$	the plane $x - 2y + 3z = 0$
(b)	$x = -5t,\ \ y = -t,\ \ z = t$	the line through $\mathbf{0}$ along $(-5, -1, 1)$
(c)	only $x = y = z = 0$	the zero subspace $\{\mathbf{0}\}$
(d)	every (x, y, z)	all of $\mathbb{R}^3$

> A plane, a line, the origin, all of space: every entry of Table 1 for $\mathbb{R}^3$ occurs as the solution space of a homogeneous system.

THEOREM 4.2.6

Different spanning sets, same span

> **THEOREM 4.2.6** $\text{span}\{\mathbf{v}_1, \dots, \mathbf{v}_r\} = \text{span}\{\mathbf{w}_1, \dots, \mathbf{w}_k\}$ if and only if each $\mathbf{v}_i$ is a linear combination of $\mathbf{w}_1, \dots, \mathbf{w}_k$, and each $\mathbf{w}_j$ is a linear combination of $\mathbf{v}_1, \dots, \mathbf{v}_r$.

Why: if every $\mathbf{v}_i$ lies in the subspace $\text{span}\{\mathbf{w}_1, \dots, \mathbf{w}_k\}$, then by Theorem 4.2.3(b) so does all of $\text{span}\{\mathbf{v}_1, \dots, \mathbf{v}_r\}$. Then swap the roles.

Example

$$\text{span}\{(1,0), (0,1)\} = \text{span}\{(1,1), (1,-1)\} = \mathbb{R}^2$$

since $(1,0) = \frac{1}{2}(1,1) + \frac{1}{2}(1,-1)$ and $(0,1) = \frac{1}{2}(1,1) - \frac{1}{2}(1,-1)$, while $(1,1)$ and $(1,-1)$ are plainly combinations of $(1,0)$ and $(0,1)$.

Two DIFFERENT sets can span the SAME subspace: a spanning set is never unique.

SUMMARY

Unit 3A in one slide

4.1 — vector spaces

- A set with an addition and a scalar multiplication that obey ten axioms: $\mathbb{R}^n$, M_{mn}, $\mathbb{R}^\infty$, $F(-\infty, \infty)$, $\{\mathbf{0}\}$. One failed axiom, shown by one specific counterexample, rules a set out. From the axioms: $0\mathbf{u} = \mathbf{0}$, $k\mathbf{0} = \mathbf{0}$, $(-1)\mathbf{u} = -\mathbf{u}$.

4.2 — subspaces

- A subset is a subspace exactly when it contains $\mathbf{0}$ and is closed under addition and scalar multiplication. In $\mathbb{R}^3$: $\{\mathbf{0}\}$, lines and planes through $\mathbf{0}$, and $\mathbb{R}^3$. Intersections of subspaces are subspaces; unions usually are not.

4.2 — two sources of subspaces for free

- $\text{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_r\}$, the smallest subspace containing the $\mathbf{v}$'s, and the solution space of $A\mathbf{x} = \mathbf{0}$. Whether $\mathbf{w}$ lies in a span, and whether vectors span $\mathbb{R}^n$, is decided by solving a linear system.