PHYSICS 501 FALL 2019
4th HOMEWORK-Solutions
Dr. V. Lempesis
Hand in: Sunday 10th of November at 23:59

1. Find the Fourier series for the function:

$$f(x)=\begin{cases} -1 & -\pi < x < -\pi/2 \\ 0 & -\pi/2 < x < \pi/2 \\ 1 & \pi/2 < x < \pi \end{cases}$$

(Hint: Be careful with the discontinuity problems. Also in your solution make a plot of the function, it will help you a lot).

(5 marks)

Solution:
The function is clearly an odd function, with period 2π, so it will contain only sin's. Thus

$$f(x)=\sum_{n=1}^{\infty} b_n \sin(nx)$$

with

$$b_n=\frac{2}{\pi}\int_0^{\pi} f(x)\sin(nx)\,dx \Rightarrow b_n=\frac{2}{\pi}\int_{\pi/2}^{\pi} f(x)\sin(nx)\,dx$$

$$b_n=\frac{2}{\pi}\int_{\pi/2}^{\pi}\sin(nx)\,dx \Rightarrow b_n=-\frac{2}{\pi n}\cos nx\Big|_{\pi/2}^{\pi}$$

$$b_n=-\frac{2}{n\pi}\left[\cos(n\pi)-\cos(n\pi/2)\right]$$

Check at discontinuities:

a) $f_-(-\pi/2)=-1$, $f_+(-\pi/2)=0$, so

$$f(-\pi/2)=\frac{1}{2}\left[f_-(-\pi/2)+f_+(-\pi/2)\right]=\frac{1}{2}\left[-1+0\right]=-\frac{1}{2}$$

Let us see what our series gives at $-\pi/2$

$$f(-\pi/2)=\sum_{n=1}^{\infty} b_n \sin\left(-n\pi/2\right)=\sum_{n=1}^{\infty}\frac{2}{n\pi}\left[\cos\left(n\pi/2\right)-\cos\left(n\pi\right)\right]\sin\left(-n\pi/2\right)=$$

$$\frac{2}{\pi}\left[\cos\left(\pi/2\right)-\cos\left(\pi\right)\right]\sin\left(-\pi/2\right)+\frac{2}{2\pi}\left[\cos\left(2\pi/2\right)-\cos\left(2\pi\right)\right]\sin\left(-2\pi/2\right)$$

$$+\frac{2}{3\pi}\left[\cos\left(3\pi/2\right)-\cos\left(3\pi\right)\right]\sin\left(-3\pi/2\right)+\frac{2}{4\pi}\left[\cos\left(4\pi/2\right)-\cos\left(4\pi\right)\right]\sin\left(-4\pi/2\right)+$$

$$+\frac{2}{5\pi}\left[\cos\left(5\pi/2\right)-\cos\left(5\pi\right)\right]\sin\left(-5\pi/2\right)...=$$

$$-\frac{2}{\pi}+0+\frac{2}{3\pi}+0-\frac{2}{5\pi}+...=$$

$$-\frac{2}{\pi}\left(1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+....\right)=-\frac{2}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}=-\frac{2}{\pi}\frac{\pi}{4}=-\frac{1}{2}$$

b) The same procedure has to be done at $\pi/2$

So our value obtained from the series agrees at discontinuity with the value got from

$$f(-\pi/2)=\frac{1}{2}\left[f_-(-\pi/2)+f_+(-\pi/2)\right]$$

Vasileios Lembessis 15/11/2019 10:31

Comment [1]: If you do not know That the value of the series is $\pi/4$. This is not a problem. At least you should have done the process correctly up to this point.

2. In right circular cylindrical coordinates a particular vector function is given by $\mathbf{V}\left(\rho,\varphi\right)=\hat{\rho}_0 V_\rho\left(\rho,\varphi\right)+\hat{\varphi}_0 V_\varphi\left(\rho,\varphi\right)$. Show that $\vec{\nabla}\times\mathbf{V}$ has only a z-component.

(5 marks)

Solution:

$$\vec{\nabla}\times\mathbf{V}\left(\rho,\varphi\right)=\frac{1}{\rho}\begin{vmatrix}\hat{\rho}_0 & \rho\hat{\varphi}_0 & \hat{k}\\ \partial/\partial\rho & \partial/\partial\phi & \partial/\partial z\\ V_\rho\left(\rho,\varphi\right) & \rho V_\varphi\left(\rho,\varphi\right) & 0\end{vmatrix}=$$

$$\frac{1}{\rho}\left\{\hat{\rho}_0\left[0-\frac{\partial V_\phi\left(\rho,\phi\right)}{\partial z}\right]-\rho\left[0-\frac{\partial V_\rho\left(\rho,\phi\right)}{\partial z}\right]\hat{\varphi}_0+\left[\frac{\partial\rho V_\phi\left(\rho,\phi\right)}{\partial\rho}-\frac{\partial V_\rho\left(\rho,\phi\right)}{\partial\phi}\right]\hat{k}\right\}=$$

$$\frac{1}{\rho}\left[\frac{\partial\rho V_\phi\left(\rho,\phi\right)}{\partial\rho}-\frac{\partial V_\rho\left(\rho,\phi\right)}{\partial\phi}\right]\hat{k}$$

3. A calculation of the magneto-hydrodynamics pinch effect involves the evaluation of $\left(\mathbf{B}\cdot\vec{\nabla}\right)\mathbf{B}$. If the magnetic induction $\mathbf{B}$ is taken to be $\mathbf{B}=-\hat{\varphi}_0 B_\varphi\left(\rho\right)$, show that $\left(\mathbf{B}\cdot\vec{\nabla}\right)\mathbf{B}=-\hat{\rho}_0 B_\varphi^2/\rho$.

(5 marks)

Solution:

Let $\mathbf{B}=-\hat{\varphi}_0 B_\varphi\left(\rho\right)$, so $B_z=B_\rho=0$.

$$\mathbf{B}\cdot\vec{\nabla}=\left(-\hat{\varphi}_0 B_\varphi\left(\rho\right)\right)\cdot\left(\hat{\rho}_0\frac{\partial}{\partial\rho}+\hat{\varphi}_0\frac{1}{\rho}\frac{\partial}{\partial\phi}+\hat{k}\frac{\partial}{\partial z}\right)=-\frac{1}{\rho}B_\varphi\left(\rho\right)\frac{\partial}{\partial\phi}$$

$$\left(\mathbf{B}\cdot\vec{\nabla}\right)\cdot\mathbf{B}=-\frac{1}{\rho}B_\varphi\left(\rho\right)\frac{\partial}{\partial\phi}\left[-\hat{\varphi}_0 B_\varphi\left(\rho\right)\right]=\frac{1}{\rho}B_\varphi\left(\rho\right)\left[\frac{\partial\hat{\varphi}_0}{\partial\phi}B_\varphi\left(\rho\right)+\hat{\varphi}_0\frac{\partial B_\varphi\left(\rho\right)}{\partial\phi}\right]$$

$$=\frac{1}{\rho}B_\varphi^2\left(\rho\right)\frac{\partial\hat{\varphi}_0}{\partial\phi}=-\frac{1}{\rho}B_\varphi^2\left(\rho\right)\hat{\rho}_0$$

4. Working in spherical coordinates prove that:

$$\vec{\nabla}\cdot\hat{\mathbf{r}}f(r)=\frac{2f(r)}{r}+\frac{df}{dr}$$

(5 marks)

Solution:

$$\vec{\nabla}\cdot\hat{\mathbf{r}}f(r)=\left(\hat{\mathbf{r}}\frac{\partial}{\partial r}+\hat{\theta}_0\frac{\partial}{r\partial\theta}+\hat{\phi}_0\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\right)\cdot\left(\hat{\mathbf{r}}f(r)\right)=$$

$$\hat{\mathbf{r}}\cdot\left[\frac{\partial}{\partial r}\left(\hat{\mathbf{r}}f(r)\right)\right]+\hat{\theta}_0\cdot\left[\frac{\partial}{r\partial\theta}\left(\hat{\mathbf{r}}f(r)\right)\right]+\frac{1}{r\sin\theta}\hat{\phi}_0\cdot\left[\frac{\partial}{\partial\phi}\left(\hat{\mathbf{r}}f(r)\right)\right]=$$

$$\hat{\mathbf{r}}\cdot\left[\frac{\partial f(r)}{\partial r}\hat{\mathbf{r}}+f(r)\frac{\partial\hat{\mathbf{r}}}{\partial r}\right]+\frac{1}{r}\hat{\theta}_0\cdot\left[\frac{\partial f(r)}{\partial\theta}\hat{\mathbf{r}}+f(r)\frac{\partial\hat{\mathbf{r}}}{\partial\theta}\right]+\frac{1}{r\sin\theta}\hat{\phi}_0\cdot\left[\frac{\partial f(r)}{\partial\phi}\hat{\mathbf{r}}+f(r)\frac{\partial\hat{\mathbf{r}}}{\partial\phi}\right]=$$

$$\left(\hat{\mathbf{r}}\cdot\hat{\mathbf{r}}\right)\frac{df(r)}{dr}+\frac{f(r)}{r}\hat{\theta}_0\cdot\left(\frac{\partial\hat{\mathbf{r}}}{\partial\theta}\right)+\frac{f(r)}{r\sin\theta}\hat{\phi}_0\cdot\left(\frac{\partial\hat{\mathbf{r}}}{\partial\phi}\right)$$

But

$$\left(\hat{\mathbf{r}}\cdot\hat{\mathbf{r}}\right)=1,\quad\left(\frac{\partial\hat{\mathbf{r}}}{\partial\theta}\right)=\hat{\theta}_0,\quad\left(\frac{\partial\hat{\mathbf{r}}}{\partial\phi}\right)=\hat{\phi}_0\sin\theta$$

Thus

$$\vec{\nabla}\cdot\hat{\mathbf{r}}f(r)=\frac{2f(r)}{r}+\frac{df}{dr}$$