Section 4.8

Antiderivatives

Recovering a function from its derivative

Math 111 · Integral Calculus

Thomas' Calculus: Early Transcendentals, 15th ed. (Global Edition)

Department of Mathematics, King Saud University

Where This Lecture Fits

Chapters 4 and 5 — Applications of Derivatives and Integrals

5.2	→	5.3	→	4.8	→	5.4
Sigma Notation and Finite Sums		The Definite Integral		Antiderivatives		Fundamental Theorem of Calculus

✓ *done*

today

Why antiderivatives, and why now?

In 5.3 we defined the definite integral as a limit of Riemann sums — correct, but painful to compute. Antiderivatives are the missing link: in 5.4 the Fundamental Theorem will turn every one of those limits into a simple evaluation. This is why antidifferentiation is also called integration.

∫ The Idea: Run Differentiation Backwards

FORWARD

Differentiation

$$x^3 \longrightarrow 3x^2$$

Given F, find F′. Familiar from Chapter 3.

BACKWARD

Antidifferentiation

$$3x^2 \longrightarrow x^3$$

Given f, find F with F′ = f. This is §4.8.

The question to ask yourself every time:

“What function do I know whose derivative is this?”

Every answer can be checked immediately — just differentiate it and compare.

Definition

DEFINITION

A function ***F*** is an **antiderivative** of ***f*** on an interval ***I*** if $F'(x) = f(x)$ for all x in I.

1 Notation

Capital letters: F for an antiderivative of f, G for g, H for h.

2 On an interval

The interval I matters. $F' = f$ must hold at every point of I.

3 Not unique

If F works, so does F + 1, F − 7, F + C … more on the next slide.

Example 1 — Find an antiderivative for each function

(a)	$f(x) = 2x$	$\longrightarrow$ $\mathbf{F(x) = x^2}$	$F' = 2x$ ✓
(b)	$g(x) = \cos x$	$\longrightarrow$ $\mathbf{G(x) = \sin x}$	$G' = \cos x$ ✓
(c)	$h(x) = \sec^2 x$	$\longrightarrow$ $\mathbf{H(x) = \tan x}$	$H' = \sec^2 x$ ✓

Every answer above was found by asking "what has this as its derivative?" and then confirmed by differentiating. Do the same on every exercise.

∫ How Many Antiderivatives Are There?

$F(x) = x^2$ is not the only function with derivative $2x$.

So do $x^2 + 1$, $x^2 - 5$, $x^2 + C$ for every constant C.

By Corollary 2 of the Mean Value Theorem (§4.2), any two antiderivatives of f differ by a constant — so there are no others.

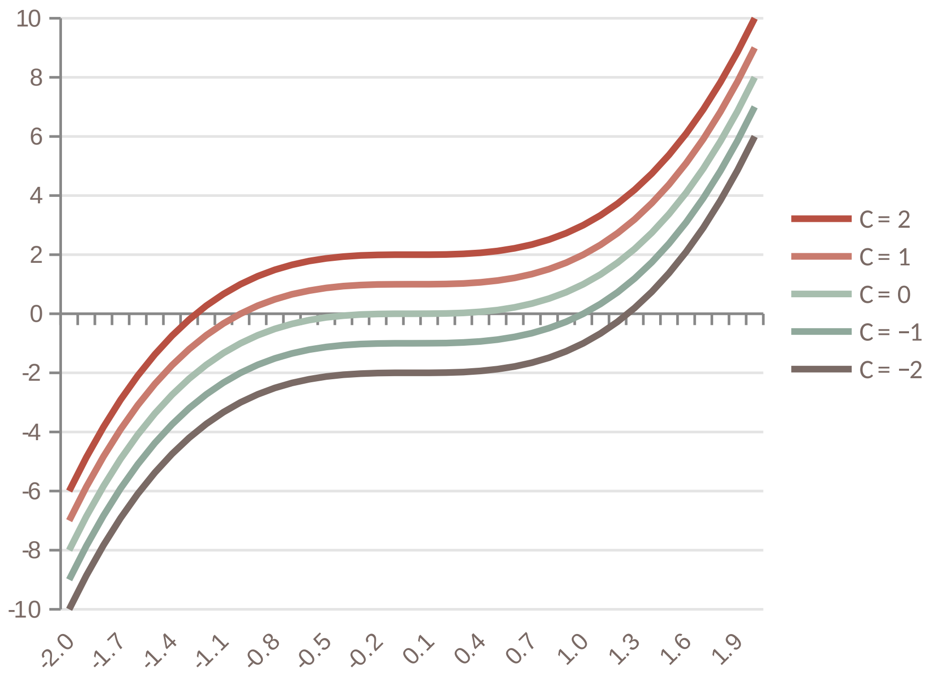
THEOREM 8

If F is an antiderivative of f on an interval I, then the most general antiderivative of f on I is

$$F(x) + C$$

where C is an arbitrary constant.

The family $y = x^3 + C$



Vertical translations of one another — they fill the plane without overlapping.

Example 2 — Picking one member of the family

Find an antiderivative of $f(x) = 3x^2$ that satisfies $F(1) = -1$.

1. **General antiderivative**

   Since $(x^3)' = 3x^2$, $F(x) = x^3 + C$ gives all antiderivatives of f.

2. **Use the condition**

   $F(1) = (1)^3 + C = 1 + C$. Set $1 + C = -1$, so $C = -2$.

3. **Answer**

   $F(x) = x^3 - 2$ — the curve of the family passing through (1, −1).

A condition of the form $F(x_0) = y_0$ always determines C — and therefore selects exactly one curve.

Table 4.2 — Antiderivative Formulas (k a nonzero constant)

Read every row backwards: differentiate the right column and you get the left one.

Function	General antiderivative
x^n	$x^{n+1}/(n+1) + C, \quad n \neq -1$
$\sin kx$	$-(1/k) \cos kx + C$
$\cos kx$	$(1/k) \sin kx + C$
$\sec^2 kx$	$(1/k) \tan kx + C$
$\csc^2 kx$	$-(1/k) \cot kx + C$
$\sec kx \tan kx$	$(1/k) \sec kx + C$
$\csc kx \cot kx$	$-(1/k) \csc kx + C$

Check one to see the pattern: $d/dx\,[\,(1/k) \tan kx + C\,] = (1/k) \cdot k \sec^2 kx = \sec^2 kx$ — true for every C and every $k \neq 0$.

Table 4.2 — Inverse Trigonometric Rows

Listed for completeness — we return to these in §3.9.

Function	General antiderivative
$1/\sqrt{(1 - k^2x^2)}$	$(1/k) \arcsin kx + C$
$1/(1 + k^2x^2)$	$(1/k) \arctan kx + C$
$1/(x\sqrt{(k^2x^2 - 1)})$	$\text{arcsec } kx + C, \quad kx > 1$

For now: just recognise these three forms. The inverse trigonometric functions themselves are studied in §3.9, so you will not be asked to derive them before then.

Example 3 — Using the table

Find the general antiderivative of each function.

(a) $f(x) = x^5$ *Formula 1, n = 5*

$F(x) = x^6/6 + C$

(b) $g(x) = 1/\sqrt{x}$ *Formula 1, n = −½*

$G(x) = 2\sqrt{x} + C$

(c) $h(x) = \sin 2x$ *Formula 2, k = 2*

$H(x) = -(\cos 2x)/2 + C$

(d) $i(x) = \cos(x/2)$ *Formula 3, k = ½*

$I(x) = 2\sin(x/2) + C$

In each case: rewrite the function so it matches a line of Table 4.2, read off the antiderivative, then differentiate to check.

Table 4.3 — Linearity Rules

Rule	Function	General antiderivative
Constant Multiple	$k\,f(x)$	$k\,F(x) + C$, k a constant
Sum or Difference	$f(x) \pm g(x)$	$F(x) \pm G(x) + C$

Example 4 — Find the general antiderivative of $f(x) = 3/\sqrt{x} + \sin 2x$

From Example 3(b): $G(x) = 2\sqrt{x}$ is an antiderivative of $1/\sqrt{x}$, so $3G(x) = 6\sqrt{x}$ handles the first term.

From Example 3(c): $H(x) = -(1/2)\cos 2x$ is an antiderivative of $\sin 2x$.

Sum Rule: $F(x) = 3G(x) + H(x) + C = 6\sqrt{x} - (1/2)\cos 2x + C$

One constant C at the end — not one per term.

Indefinite Integrals — the notation

DEFINITION

The collection of all antiderivatives of f is called the indefinite integral of f with respect to x, denoted $\int f(x)\,dx$

$$\underbrace{\int}_{\text{integral sign}} \quad \underbrace{f(x)}_{\text{integrand}} \quad \underbrace{dx}_{\text{variable of integration}}$$

Example 1 rewritten in this notation: $\int 2x\,dx = x^2 + C$ $\int \cos x\,dx = \sin x + C$

The integrand is always followed by a differential telling you the variable of integration.

Example 6 — Evaluate $\int (x^2 - 2x + 5)\,dx$

TERM BY TERM

$$\int x^2\,dx - 2\int x\,dx + 5\int dx$$

$$= (x^3/3 + C_1) - 2(x^2/2 + C_2) + 5(x + C_3)$$

$$= x^3/3 - x^2 + 5x + (C_1 - 2C_2 + 5C_3)$$

$$= x^3/3 - x^2 + 5x + C$$

RECOMMENDED

Find the simplest antiderivative for each term, then add one arbitrary constant at the very end:

$$\int (x^2 - 2x + 5)\,dx$$

$$= x^3/3 - x^2 + 5x + C$$

Common mistake: writing a separate constant for every term. Any combination $C_1 - 2C_2 + 5C_3$ is again just an arbitrary constant, so one C is enough — and always check by differentiating the answer.

Table 4.4 — Basic Integration Formulas

The same table as 4.2, written with the integral sign. Learn this list.

Integral	Result	Integral	Result
$\int x^n \, dx$	$x^{n+1}/(n+1) + C,\ n \neq -1$	$\int \sec x \tan x \, dx$	$\sec x + C$
$\int \sin x \, dx$	$-\cos x + C$	$\int \csc x \cot x \, dx$	$-\csc x + C$
$\int \cos x \, dx$	$\sin x + C$	$\int dx/\sqrt{(1 - x^2)}$	$\arcsin x + C$
$\int \sec^2 x \, dx$	$\tan x + C$	$\int dx/(1 + x^2)$	$\arctan x + C$
$\int \csc^2 x \, dx$	$-\cot x + C$	$\int dx/(x\sqrt{(x^2 - 1)})$	$\operatorname{arcsec} x + C,\ x > 1$

The three inverse trigonometric rows are quoted here and established later, in §3.9.

Summary

1. **F is an antiderivative of f on I** — means $F'(x) = f(x)$ for every x in I.
2. **Theorem 8** — the general antiderivative is $F(x) + C$ — a whole family of curves.
3. **Choosing one member** — a condition such as $F(x_0) = y_0$ fixes C and selects a single curve.
4. **Indefinite integral** — $\int f(x)\,dx$ denotes all antiderivatives of f. One constant C, added at the end.
5. **Always check** — differentiate your answer and compare with the integrand.

Practice

Exercises 4.8

1–24 finding antiderivatives

25–70 indefinite integrals

71–82 verifying formulas

Do as many as you can mentally, and check every answer by differentiating.

Next lecture: §5.4 — The Fundamental Theorem of Calculus