Final Exam – Semester I, 1447

Department of Mathematics, College of Science, KSU

Course: Math 481 Total Marks: 40 Duration: 3 Hours

Question 1 [6 points]

(a) Let $f : [a, b] \to \mathbb{R}$ be an increasing function. Prove that f is Riemann integrable on $[a, b]$. [2]

(b) Using the definition of the Riemann integral as the limit of Riemann sums, evaluate

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{k^2}{n^3 + k^3}.$$

[4]

Question 2 [8 points]

(a) For each $n \in \mathbb{N}$, define the function

$$f_n(x) = \frac{\sin(nx)}{nx}, \qquad x \in (0, \infty).$$

(i) Find the pointwise limit of the sequence (f_n) on $(0, \infty)$. [2]

(ii) Determine whether (f_n) converges uniformly on $(0, \infty)$, and justify your answer. [2]

(iii) Determine whether (f_n) converges uniformly on $[a, \infty)$ for any $0 < a$. Justify your answer. [2]

(b) Justify the interchange of summation and integration, and evaluate

$$\int_0^{\pi} \left(\sum_{n=1}^{\infty} \frac{\sin(nx)}{n^3} \right) dx.$$

[2]

Question 3 [8 points]

(a) Consider the series of functions

$$\sum_{n=1}^{\infty} \frac{nx}{1+n^3x^3}, \qquad x \geq 0.$$

Show that this series does not converge uniformly on $[0, \infty)$, but does converge uniformly on every closed interval $[a, b] \subset (0, \infty)$. [5]

(b) Using part (a), prove that the sum function

$$f(x) = \sum_{n=1}^{\infty} \frac{nx}{1+n^3x^3}$$

is continuous on $(0, \infty)$. [3]

Question 4 [8 points]

(i) (a) Define what it means for a set $E \subset \mathbb{R}$ to be *Lebesgue measurable* in terms of the Lebesgue outer measure m^*.

(b) Prove that if a set $E \subset \mathbb{R}$ satisfies

$$m^*(E) = 0,$$

then both E and its complement E^c are Lebesgue measurable.

[5]

(ii) Give a concrete example of a function that is Lebesgue integrable but not Riemann integrable on $[0, 1]$, and briefly explain why it satisfies these properties. [3]

Question 5 [10 points]

(a) Let $(f_n)_{n\geq 1}$ be the sequence of functions

$$f_n(x) = xe^{-nx}, \qquad x \in [0, \infty).$$

(i) Show that each f_n is Lebesgue measurable and compute

$$\int_0^{\infty} f_n(x)\, dx.$$

[3]

xe^{-2x}

$\frac{1}{2}\int_0^{\infty} e^{-2x}$

$-\frac{1}{4}(0-1) = \frac{1}{4}$

(ii) Show that the series

$$\sum_{n=1}^{\infty} f_n(x)$$

converges *pointwise* for all $x > 0$. Then, justify the interchange of summation and integration by applying the *Monotone Convergence Theorem*, verifying its hypotheses. Finally, conclude that

$$\int_0^{\infty} \left(\sum_{n=1}^{\infty} f_n(x) \right) dx = \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

[5]

(b) Let

$$g_n(x) = \frac{(\sin x)^n}{1 + x^2}, \qquad x \geq 0.$$

(i) Show that (g_n) converges pointwise to 0 *almost everywhere* on $[0, \infty)$.

(ii) Using the Dominated Convergence Theorem, prove that

$$\lim_{n \to \infty} \int_0^{\infty} g_n(x)\, dx = 0.$$

[2]

لا يكتب في
هذا الهامش

$\boxed{Q1}$ a) ~~Since f is an increase~~

let f be an increasing function

if $f(a) = f(b) \Rightarrow f$ is a constant function

hence it is Riemann integrable

if $f(a) \neq f(b)$, ~~given $\varepsilon>0$~~

let P be a partition $P = \{x_0, x_1, \dots, x_n\}$

given $\varepsilon > 0$ let $\|P\| < \frac{\varepsilon}{f(b) - f(a)}$

2

since f is increasing

$M_i = f(x_{i+1})$, $m_i = f(x_i)$, on $[x_i, x_{i+1}]$

$$U(f,P) - L(f,P) = \sum_{i=0}^{n-1} (M_i - m_i) \Delta x_i$$

$$= \sum_{i=0}^{n-1} (f(x_{i+1}) - f(x_i)) \Delta x_i$$

$$< \|P\| \sum_{i=0}^{n-1} (f(x_{i+1}) - f(x_i))$$

$$< \frac{\varepsilon}{f(b) - f(a)} (f(b) - f(a)) = \varepsilon$$

by Riemann's criterion

f is integrable

(Riemann integrable)

لا يكتب في
هذا الهامش

$\boxed{Q1}$ (b) $\lim_{n\to\infty} \sum_{K=1}^{n} \frac{K^2}{n^3+K^3}$

$$\lim_{n\to\infty} \sum_{K=1}^{n} \frac{K^2}{n^3+K^3} = \lim_{n\to\infty} \sum_{K=1}^{n} \frac{\frac{K^2}{n^3}}{\frac{n^3}{n^3}+\frac{K^3}{n^3}} = \lim_{n\to\infty} \frac{1}{n} \sum_{K=1}^{n} \frac{(\frac{K}{n})^2}{1+(\frac{K}{n})^3}$$

$\Delta x_k = \frac{1}{n}$, $x_k = \frac{K}{n}$, $K = 1, 2, \dots K$

$b - a = 1$, $a = 0 \Rightarrow b = 1$

$f(x) = \frac{x^2}{1+x^3}$

by Riemann sum

$$\lim_{n\to\infty} \sum_{K=1}^{n} \frac{K^2}{n^3+K^3} = \int_0^1 \frac{x^2}{1+x^3}\,dx = \frac{1}{3}\int_0^1 \frac{3x^2}{1+x^3}\,dx$$

$$= \frac{1}{3}\left[\ln|1+x^3|\right]_0^1 = \frac{1}{3}(\ln(2) - \ln(1))$$

$$= \frac{1}{3}\ln(2)$$

4

$x^2(1+x^3)^{-1}$

$\boxed{Q2}$ a) $f_n(x) = \frac{\sin(nx)}{nx}$, $x \in (0, \infty)$

(i) the Pointwise limit

~~$\lim_{n\to\infty} \left|\frac{\sin(nx)}{nx}\right|$~~

$\left|\frac{\sin(nx)}{nx}\right| \le \frac{1}{nx}$, $\lim_{n\to\infty} \frac{1}{nx} \to 0$

So the P.W limit for $\frac{\sin(nx)}{nx}$ is 0 on $(0, \infty)$

2

لا يكتب في
هذا الهامش

(ii) let $x_n = \frac{\pi}{2n}$, $x_n \in (0,\infty)$

$$f_n\left(\frac{\pi}{2n}\right) = \frac{\sin\left(n\frac{\pi}{2n}\right)}{n\frac{\pi}{2n}} = \frac{2\sin\left(\frac{\pi}{2}\right) n}{n\pi} = \frac{2}{\pi}$$

$$\sup_{x\in(0,\infty)} |f_n(x) - f(x)| = \sup_{x\in(0,\infty)} |f_n(x)| \geq \frac{2}{\pi} \not\to 0$$

2

by sup-norm criterion f_n doesn't converge uniformly on $(0,\infty)$

(iii)

on $[a,\infty)$ $a>0$

$$\sup_{x\in[a,\infty)} |f_n(x) - f(x)| = \sup_{x\in[a,\infty)} \left|\frac{\sin(nx)}{nx}\right|$$

$$= \sup_{x\in[a,\infty)} \frac{|\sin(nx)|}{nx} \leq \frac{1}{nx} \leq \frac{1}{na} \to 0$$

2

$a \leq x$

$na \leq nx$

$\frac{1}{na} \geq \frac{1}{nx}$

by sup-norm criterion f_n converges uniformly to f on $[a,\infty)$, $a>0$.

لا يكتب في
هذا الهامش

$\boxed{Q2}$ (b) $\int_0^{\pi} \left(\sum_{n=1}^{\infty} \frac{\sin(nx)}{n^3} \right) dx$

2

$f_n(x) = \frac{\sin(nx)}{n^3}$

$\left| \frac{\sin(nx)}{n^3} \right| \leq \frac{1}{n^3}$

$\sum_{n=1}^{\infty} \frac{1}{n^3}$ converges by P-test $(P = 3 > 1)$

By M-test $\sum_{n=1}^{\infty} \frac{\sin(nx)}{n^3}$ converges uniformly

thus $\int_0^{\pi} \sum_{n=1}^{\infty} f_n(x) \, dx = \sum_{n=1}^{\infty} \int f_n(x) \, dx$

2

$\int_0^{\pi} \left(\sum_{n=1}^{\infty} \frac{\sin(nx)}{n^3} \right) dx = \sum_{n=1}^{\infty} \frac{1}{n^3} \int_0^{\pi} \sin(nx) \, dx$

$\int_0^{\pi} \sin(nx) \, dx = \frac{-1}{n} (\cos(nx)) \Big]_0^{\pi} = \frac{-1}{n} ((-1)^n - 1)$

$= \frac{1}{n} ((-1)^{n+1} + 1)$

the sum becomes $\sum_{n=1}^{\infty} \frac{1}{n^4} (1 + (-1)^{n+1})$

if n is even $1 + (-1)^{n+1} = 0$

if n is odd $1 + (-1)^{n+1} = 2$

~~the sum~~ $\Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^4} (1 + (-1)^{n+1}) = \sum_{n=1}^{\infty} \frac{2}{(2n-1)^4}$

لا يكتب في هذا الهامش

$\boxed{Q3}$ (a) $\sum_{n=1}^{\infty} \frac{nx}{1+n^3x^3}$, $x \geq 0$

$f_n(x) = \frac{nx}{1+n^3x^3}$ on $[0,\infty)$

take $x_n = \frac{1}{n}$, $f(\frac{1}{n}) = \frac{1}{2}$

$\sup_{x\in[0,\infty)} |f_n(x)| \geq \frac{1}{2} \not\longrightarrow 0$

So $\sum_{n=1}^{\infty} \frac{nx}{1+n^3x^3}$ doesn't cv uniformly on $[0,\infty)$

$a \leq x \leq b$

$nx \leq b$

~~on every bounded interval~~

on every closed interval $[a,b] \subseteq (0,\infty)$

$a \leq x$

$n^3a \leq n^3x$

$1+n^3a \leq n^3a$

$f_n(x) = \frac{nx}{1+n^3x^3}$

$a > b$

$a^3 > b^3$

$\left|\frac{nx}{1+n^3x^3}\right| = \frac{nx}{1+n^3x^3}$ ~~$\frac{nb}{1+n^3a^3} \leq \frac{nb}{n^3a^3} = \frac{b}{a^3}\frac{1}{n^2}$~~

$\leq \frac{nx}{n^3x^3} = \frac{1}{n^2x^2} \leq \frac{1}{n^2a^2}$

5

~~$x \leq b \Rightarrow nx \leq nb$~~

~~$x \geq a > 0 \Rightarrow x^3 \geq a^3 \Rightarrow n^3x^3 \geq n^3a^3 \Rightarrow 1+n^3x^3 \geq 1+n^3a^3$~~

$x \geq a > 0 \Rightarrow x^2 \geq a^2 \Rightarrow n^2x^2 \geq n^2a^2$ ~~$\frac{1}{1+n^3x^3} \leq \frac{1}{1+n^3a^3}$~~

~~$\frac{nb}{1+n^3a^3} \leq \frac{nb}{n^3a^3} = \frac{b}{a^3}\frac{1}{n^2}$~~ $\Rightarrow \frac{1}{n^2x^2} \leq \frac{1}{n^2a^2}$

$\frac{1}{a^2}\sum \frac{1}{n^2}$ cv by P-test (P=2)

By M-test $\sum_{n=1}^{\infty} \frac{nx}{1+n^3x^3}$ converges uniformly

on $[a,b]$

لا يكتب في
هذا الهامش

$\boxed{Q3}$ (b) by Part (a)

$f(x) = \sum_{n=1}^{\infty} \frac{nx}{1+n^2x^3}$ converges uniformly

on every interval $[a,b]$, $0<a\le b<\infty$

$f_n(x) = \frac{nx}{1+n^3x^3}$ is continuous for every

$x \in (0,\infty)$, for all n

~~[illegible]~~

from ~~[illegible]~~ continuity theorem

for series

3

$\sum_{n=1}^{\infty} \frac{nx}{1+n^3x^3}$ converges to a continous

f on $[a,b]$, ~~[illegible]~~ $0<a\le b<\infty$

~~for~~ since a and b are arbitrary

let $a \to 0^+$ and $b \to \infty$

hence f is continuous on $(0,\infty)$

لا يكتب في
هذا الهامش

(i) (a)

Q4 a set $E \subseteq \mathbb{R}$ is said to be ~~measurable if~~ Lebesgue measurable if for every $A \subseteq \mathbb{R}$

$m^*(A) = m^*(A \cap E) + m^*(A \cap E^c)$

(i) (b)

if $E \subset \mathbb{R}$ S.T. $m^*(E) = 0 \Rightarrow E \in \mathcal{M}$

Proof:

$\forall A \subseteq \mathbb{R}$ since $A \cap E \subseteq E$

$0 \leq m^*(A \cap E) \leq m^*(E) = 0 \Rightarrow m^*(A \cap E) = 0$

and since $A \cap E^c \subseteq A$

$m^*(A \cap E^c) \leq m^*(A)$

hence $m^*(A \cap E) + m^*(A \cap E^c) \leq m^*(A)$ ①

the other inequality is satisfied since

$A \subseteq (A \cap E) \cup (A \cap E^c) \Rightarrow m^*(A) \leq m^*(A \cap E) + m^*(A \cap E^c)$ ②

from 1 and 2 $m^*(A) = m^*(A \cap E) + m^*(A \cap E^c)$

hence E is measurable

Since $\mathcal{M}$ is σ-algebra it is closed under compliments $\Rightarrow E^c$ is measurable

5

لا يكتب في
هذا الهامش

$\boxed{Q4}$ (ii)

take the function

$$f(x) = \begin{cases} 1 & , x \in [0,1] \cap \mathbb{Q} \\ 0 & , x \in [0,1] \cap \mathbb{Q}^c \end{cases} \quad , x \in [0,1]$$

~~this f is not Riemann integrable~~

since it is discontinuous on every

point $x \in [0,1]$

$m([0,1]) = 1 \neq 0$ ~~thus~~

hence, f is not Riemann integrable

it is lebesgue though

since f is a non-negative simple function

and $\int_{[0,1]} f \, dm = m([0,1] \cap \mathbb{Q})(1) + m([0,1] \cap \mathbb{Q}^c)(0)$

$= (0)(1) + (1)(0) = 0$

3

~~Q4~~ $\boxed{Q5}$ (a) $f_n(x) = x e^{-xn}$, $x \in [0,\infty)$

لا يكتب في
هذا الهامش

(i) $\forall n \in \mathbb{N}$

f_n is continuous hence measurable on $[0,\infty)$

$$\int_0^\infty f_n(x)\,dx = \int_0^\infty x e^{-xn}\,dx$$

let $u = x$ $\quad$ $dv = e^{-nx}\,dx$

$du = dx$ $\quad$ $v = \frac{-e^{-nx}}{n}$

3

$$\int_0^\infty x e^{-nx}\,dx = \frac{-e^{-nx}}{n} x \Big|_0^\infty + \frac{1}{n}\int_0^\infty e^{-nx}\,dx$$

$$= \lim_{t\to\infty}\left(\frac{-t}{n e^{nt}}\right) - 0 + \frac{1}{n}\int_0^\infty e^{-nx}\,dx$$

$$\lim_{t\to\infty} \frac{-t}{n e^{nt}} = \frac{-\infty}{\infty} \Rightarrow \lim_{t\to\infty} \frac{-1}{n^2 e^{nt}} = 0$$

$$\Rightarrow \int_0^\infty x e^{-nx}\,dx = \frac{1}{n}\int_0^\infty e^{-nx}\,dx$$

$$= \frac{-1}{n^2}\left[e^{-nx}\right|_0^\infty$$

$$= \frac{-1}{n^2}(0-1) = \frac{1}{n^2}$$

لا يكتب في
هذا الهامش

$\boxed{Q5}$ (a)(ii) $\sum_{n=1}^{\infty} f_n(x)$, $f_n(x) = xe^{-nx}$

$\sum_{n=1}^{\infty} xe^{-nx}$ on closed interval $[a,b]$, $0<a<b$

$|xe^{-nx}| \leq b\, e^{-na}$,

$$\begin{array}{|c|} x \geq a \\ nx \geq na \\ e^{nx} \geq e^{na} \\ e^{-nx} \leq e^{-na} \end{array}$$

$\sum_{n=1}^{\infty} b e^{-na} = b \sum_{n=1}^{\infty} \frac{1}{e^{na}}$

$= b \sum_{n=1}^{\infty} \left(\frac{1}{e^{a}}\right)^{n} = b \frac{1}{1-e^{a}} < \infty$ $\left\{ \begin{array}{l} e^{a} > 1 \\ \frac{1}{e^{a}} < 1 \end{array} \right\}$

$\cancel{x \geq a}$
$\cancel{nx \geq na}$
$\cancel{-nx \leq -na}$
$\cancel{e^{-nx} \leq e^{-na}}$

$\cancel{x \geq a}$
$\cancel{nx \geq na}$
$\cancel{e^{nx} \geq e^{na}}$
$\cancel{e^{-nx} \leq e^{-na}}$
$\cancel{\sum e^{-n}}$

By M-test $\sum_{n=1}^{\infty} xe^{-nx}$ converges uniformly on $[a,b]$

Since a,b are arbitrary $0<a\leq b$

hence it converges Pointwise on $(0,\infty)$

Let $\cancel{g_n(x) = \sum_{n=1}^{\infty} f_n(x)}$ $g_n(x) = \sum_{k=1}^{n} f_k(x)$

$f_k(x)$ is ~~increa~~ Positive $\forall n \in \mathbb{N}$

hence g_n is an increasing sequence of measurable functions

and converges Pointwise on $(0,\infty)$

$\cancel{\int_0^{\infty} g_n \, dx = \sum_{n=1}^{\infty} \int_0^{\infty} g_n(x)\, dx = \sum_{n=1}^{\infty} \int_0^{\infty} g_n(x)\, dx}$

$\cancel{\int_0^{\infty} \lim_{n\to\infty} g_n(x)}$

لا يكتب في هذا الهامش

By MCT $\int_0^\infty \lim_{n\to\infty} g_n(x)\,dx = \lim_{n\to\infty} \int_0^\infty g_n(x)\,dx$

$= \lim_{n\to\infty} \int_0^\infty \sum_{k=1}^{n} f_k(x)\,dx$

$= \lim_{n\to\infty} \sum_{k=1}^{n} \int_0^\infty f_k(x)\,dx$

$= \sum_{k=1}^{\infty} \int_0^\infty f_k(x)\,dx$

Put $k=n$

$\Rightarrow \sum_{n=1}^{\infty} \int_0^\infty f_n(x)\,dx$

5

from $\int_0^\infty \left(\sum_{n=1}^{\infty} f_n(x)\right)dx = \sum_{n=1}^{\infty} \int_0^\infty f_n(x)\,dx$ (from previous question $\int_0^\infty f_n(x)\,dx = \frac{1}{n^2}$)

hence $= \sum_{n=1}^{\infty} \frac{1}{n^2}$,

(b) $g_n(x) = \frac{(\sin(x))^n}{1+x^2}$, $x \geq 0$

(i) $\lim_{n\to\infty} \frac{(\sin x)^n}{1+x^2} = \begin{cases} 0 & , |\sin x| < 1 \\ \frac{1}{1+x^2} & , |\sin(x)| = 1 \end{cases}$

$|\sin(x)| = 1 \Rightarrow x = \frac{\pi}{2} + n\pi$ (countable)

2

Since the points where $g_n \neq 0$ are countable

the measure for them is zero

hence, g_n converges P.W. to zero

almost everywhere on $[0,\infty)$

$\boxed{\text{Q5}}$

(b) (ii)

$g_n(x) = \frac{(\sin x)^n}{1+x^2}$, $x \geq 0$,

$\left|\frac{(\sin(x))^n}{1+x^2}\right| \leq \frac{1}{1+x^2}$

$$\int_0^\infty \frac{1}{1+x^2}\,dx = [\tan^{-1}(x)]_0^\infty = \lim_{t\to\infty}(\tan^{-1}(t) - \tan^{-1}(0))$$

$$= \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$$

By DCT

g_n is integrable and

$$\lim_{n\to\infty}\int_0^\infty g_n(x)\,dx = \int_0^\infty \lim_{n\to\infty} g_n(x)\,dx = \int_0^\infty 0\,dx = 0$$

Since $\lim_{n\to\infty} g_n(x) = 0$ a.e. on $[0,\infty)$

the case where $\lim_{n\to\infty} g_n(x) \neq 0$

has measure zero

it won't affect the value of the integral.