

# Mathematics of Finance II: Derivative securities

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# Forwards: Alternative derivation of formula

## Spot transaction

- Price agreed to.
- Price paid/received.
- Item exchanged.

## Prepaid forward contract

- Price agreed to.
- Price paid/received.
- Item exchanged in  $T$ -years.

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Forward price when the underlying asset provides a known yield  $q$ :

$$F_p(0, t, T) = S_t e^{-q(T-t)}:$$

$F_p(0, t, T)$  equals the investment required in the asset at time  $t$  (today) that will yield one unit of the asset in  $T$ -years when physical delivery occurs.

$e^{-q(T-t)}$  units of the asset will grow to  $e^{-q(T-t)} \times e^{q(T-t)} = 1$ -unit of the asset in  $T$ -years, assuming that the income provided by the asset is reinvested in the asset.

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A forward contract allows the long position to delay payment for  $T$ -years and requires the short position to delay receipt.

A forward contract has two risks: **market risk** and **credit risk**. The market risk is related with the **volatility** of the asset price. The credit risk is related with the solvency of each party.

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# Futures: Definition

A future contract is a standardized agreement in which two counterparts agree to buy/sell an asset for a specified price at a specified period.

The buyer in the future contract is said to be in long position (LP) on futures.

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## Difference between forwards and futures

Recall forward contracts are privately negotiated and are not standardized. Forward contracts are entirely flexible. Forward contracts are tailor-made contracts.

Futures contracts are standardized instruments and FC have clearing houses that guarantee the transactions, which drastically lowers the probability of default to almost never.

The specific details concerning settlement and delivery are quite distinct

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Clearing houses act as third parties to all futures and options contracts - as a buyer to every clearing member seller and a seller to every clearing member buyer.

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But, unlike forward contracts, futures contracts are marked to market daily. Consider "corresponding" forward and futures contracts:

- Same underlying asset.
- Delivery date in two days.
- The contracts are identical except:
  - i) Forward contract is settled at maturity.
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## Forward prices & futures prices

**Example:** Suppose we have for  $T = 2$ :

**Day 0:**  $G(0, 0, 2) = 20$  SAR

**Day 1:**  $G(0, 1, 2) = 10$  SAR with a 50% probability and  $G(0, 1, 2) = 30$  SAR with a 50% probability

**Day 2:**  $G(0, 2, 2) = S_2$  since the futures contract terminates.

Suppose that the interest rate is a constant 10% (effective per day).



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Since there is a 50% chance of paying interest of 1 SAR and a 50% chance of earning interest of 1 SAR, there is no expected benefit from marking to market **on day 1**.

Since futures contract offers no benefit as compared to the forward contract  $F(0, 0, T) = G(0, 0, T)$ .

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Now suppose that the interest rate is not constant.

Suppose that  $r = 12\%$  on day 1 if  $G(0, 1, 2) = 30$  SAR  
and  $r = 8\%$  on day 1 if  $G(0, 1, 2) = 10$  SAR.

If on day 1  $G(0, 1, 2) = 10$  SAR then the P&L of the buyer is  
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Now there is an expected gain from marking to market  
 $= (50\% \times 0.12 - 50\% \times 0.08) = 0.02$  SAR.

Since the futures contract offers a benefit as compared to the forward contract,  $G(0, 0, T)$  must exceed  $F(0, 0, T)$ .



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With this reasoning situations:

- 1  $G(0, 0, T) = F(0, 0, T)$  when interest rates are uncorrelated with the futures price.
- 2  $G(0, 0, T) \geq F(0, 0, T)$  when interest rates are positively correlated with the futures price.
- 3  $G(0, 0, T) \leq F(0, 0, T)$  when interest rates are negatively correlated with the futures price.



# Stock index futures contracts

- Stock index: a weighted average of the prices of a selected number of stocks.
- Underlying: the portfolio of stocks comprising the index.
- Stock index futures contracts are heavily traded
- Examples of stock indices (futures exchanges):
  - S&P/TSX Canada 60 Index (ME)
  - S&P500 Composite Index (CME)
  - NYSE Composite Index (NYFE)

# Where you buy and/or sell futures contracts

Futures are bought and sold in organized futures exchanges. The biggest future exchanges are:

- South African Futures Exchange (SAFEX)
- China Financial Futures Exchange (CFFEX)
- Shanghai Futures Exchange (SHFE)
- International Petroleum Exchange of London
- New York Mercantile Exchange
- London Metal Exchange
- Tokyo Commodity Exchange

# Where you buy and/or sell futures contracts

- Hong Kong Futures Exchange (HKFE)
- Taiwan Futures Exchange (TAIFEX)
- Turkish Derivatives Exchange (TURDEX)
- Agricultural Futures Exchange of Thailand (AFET)
- Mercado Español de Futuros Financieros (MEFF)
- ICE Futures Europe, formerly London International Financial Futures and Options Exchange (LIFFE)

# Futures

Examples of underlying assets on which futures contracts are traded.

Category	Description
Stock index	S&P 500 index, Euro Stoxx 50 index, Nikkei 225, Dow-Jones Industrials, Dax, NASDAQ, Russell 2000, S&P Sectors (healthcare, utilities, technology)
Interest rate	30-year U.S. Treasury bond, 10-year U.S. Treasury notes, Fed funds rate, Euro-Bund, Euro-Bobl, LIBOR, Euribor
Foreign exchange	Euro, Japanese yen, British pound, Swiss franc, Australian dollar, Canadian dollar, Korean won
Commodity	Oil, natural gas, gold, silver, copper, aluminum, corn, wheat, lumber, hogs, cattle, milk

# Futures

Futures transactions in the USA are regulated by the Commodity Futures Trading Commission (CFTC), an agency of the USA government.

The clearinghouse matches the purchases and the sales which take place during the day.

By matching trades, the clearinghouse never takes market risk because it always has offsetting positions with different counterpart.

By having the clearinghouse as counterpart, an individual entering a future contract does not face the possible credit risk of its counterpart.

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# Futures and hedging

An airline company may want to hedge its bets against an unexpected increase in jet fuel prices.

Its traders will therefore seek to enter into a futures contract to lock in a purchase price closer to today's prices for jet fuel.

They may buy a futures contract agreeing to buy 1 million gallons of JP-8 fuel, taking delivery 90 days in the future, at a price of 3 dollars per gallon.

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# Futures and hedging

Someone else naturally wants to ensure they have a steady market for fuel.

They also want to protect themselves against an unexpected decline in fuel prices, so they will gladly enter into either a futures contract.

In this example, both parties are hedgers, rather than speculators.

They are turning to the futures market as a way to manage their exposure to risk, rather than make money off of the deal directly.

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## Futures: Arbitrage trade

There are also people who seek to make money off of changes in the price of the contract itself, when bought or sold to other investors.

Naturally, if the price of fuel rises, the contract itself becomes more valuable, and the owner of that contract could, if it chose, sell that contract for someone else who is willing to pay more for it.

It may make sense for another airline to pay 10 cents per gallon for a contract to save 20 cents. And so there is a lively and relatively liquid market for these contracts, and they are bought and sold daily on exchanges.



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## Example: The S&P 500 Futures Contract

<b>Specifications for the S&amp;P500 index futures contract</b>	
Underlying	S&P 500 index
Where traded	Chicago Mercantile Exchange
Size	$250 \times$ S&P 500 index
Months	March, June, September, December
Trading ends	Business day prior to determination of settlement price
Settlement	Cash-settled, based up on opening price of S&P500 on third Friday of expiration month

## Example: The S&P 500 Futures Contract

The S&P 500 futures contract has the S&P 500 stock index as the underlying asset. Futures on individual stocks have recently begun trading in the United States.

The notional value, or size, of the contract is the dollar value of the assets underlying one contract. In this case it is by definition

$$250\$ \times 1300 = 325,000.12$$

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On the expiration day, the S&P 500 futures contract is marked-to-market against the actual cash index. This final settlement against the cash index guarantees that the futures price equals the index value at contract expiration.

It is easy to see why the S&P 500 is cash-settled. A physical settlement process would call for delivery of 500 shares (or some large subset thereof ) in the precise percentage they make up the S&P 500 index. This basket of stocks would be expensive to buy and sell.

Cash settlement is an inexpensive alternative.

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## Margins and Marking to Market

Let us explore the logistics of holding a futures position. Suppose the futures price is 1100 and you wish to acquire a 2.2 million US \$ position in the S&P500 index.

The notional value of one contract is  $250 \times 1100 = 275000$ : this represents the amount you are agreeing to pay at expiration per futures contract.

To go long 2.2 million USA \$ of the index, you would enter into  $2.2\text{million}/0.275\text{million} = 8$  long futures contracts. The notional value of eight contracts is  $8 \times 250 \times 1100 = 2000 \times 1100 = 2.2$  million \$.

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# Margins and Marking to Market

The margin on the S&P500 contract has generally been less than the 10% we assume in this example.

**See Excel sheets for practice**

## Example: some common futures

- 1 Crude oil futures trade in units of 1000 U.S. barrels (42,000 gallons). The underlying is a US barrel. The notional amount is 1000 barrels. The current price is \$70 per barrel.  
Hence, the current value of a future contract on crude oil is \$70000.
- 2 S&P500 future contracts trade on 250 units of the index. They are cash settled. At expiration time, instead of a sale, one of the future counterpart receive a payment according with S&P500 spot price at expiration.  
The current price of S&P500 is 1500. The current value of a future contract on S&P500 is  $(250)(1500) = \$375000$ .

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Suppose that two parties agree in a future contract for crude oil for delivery in 18 months. The contract is worth \$70000.

Usually future positions are settled into the margin account either every day or every week.

By every day we mean every day which the market is open. Let us suppose that a clearinghouse settles accounts daily.

Suppose that the annual continuously compounded interest rate is  $r$ .



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Every day, the profit or loss is calculated on the investor's futures position.

If there exists a loss, the investor's broker transfers that amount from the investor's margin account to the clearinghouse.

If a profit, the clearinghouse transfers that amount to investor's broker who then deposits it into the investor's margin account. The profit for a long position in a future contract is

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where  $M_{t-(1/365)}$  is the yesterday's balance in the margin account,  $N$  is the notional amount,  $S_t$  is the current price,  $S_{t-(1/365)}$  is the yesterday price. Hence, after the settlement, the balance in the investor's margin account is

$$M_t = M_{t-(1/365)} \times \exp(r/365) + N(S_t - S_{t-(1/365)})$$

. The profit for a short position in a future contract is

$$M_{t-(1/365)} \times (1 - \exp(r/365)) + N(S_{t-(1/365)} - S_t).$$

Marking-to-market is to calculate the value of a future contract according with the current value of the asset.

## Example: some common futures

On July 5, 2007, ABC enters a long future contract for 1,000 U.S. barrels of oil at \$71.6 per barrel.

The margin account is 50% of the market value of the futures' underlier.

The annual continuously compounded rate of return is 6%.

(i) On July 6, 2007, the price of oil is \$70.3. What is the balance in ABC's margin account after settlement?

(ii) On July 7, 2007, the price of oil is \$72.1.

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## Example: some common futures

**Solution:** (i) The initial balance in ABC's margin account is  $0.50 \times 1000 \times 71.6 = 35800$ .

The balance in ABC's margin account on July 6, 2007, after settlement, is

$$M_{t-(1/365)} \exp(r/365) + N(S_t - S_{t-(1/365)}) \\ = (35800) \exp(0.06/365) + (1000)(70.3 - 71.6) = 35105.89.$$

Since the price of the oil decreases, the value of having 1000 barrels in 18 months decreases.

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**Solution:** (ii) The balance in ABC's margin account on July 6, 2007, after settlement, is

$$\begin{aligned} & M_{t-(1/365)} \exp(r/365) + N(S_t - S_{t-(1/365)}) \\ &= (35105.89) \exp(0.06/365) + (1000)(72.1 - 70.3) \\ &= 35711.56. \end{aligned}$$

Notice that this balance is different from

$$(35800) \exp(0.06(2/365)) + (1000)(72.1 - 71.6) = 36311.77.$$

In the first day, ABC's account balance was smaller. So, ABC lost interest because the drop on price on July 6, 2007.

## Example: some common futures

If the balance in the margin account falls the clearinghouse has less protection against default.

Investors are required to keep the margin account to a minimum level. This level is a fraction of the initial margin.

The maintenance margin is the fraction of the initial margin which participants are asked to hold in their accounts.

If the balance in the margin account falls below this level, an investor's broker will require the investor to deposit funds sufficient to restore the balance to the initial margin level.

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Such a demand is called a margin call. If an investor fail to the deposit, the investor's broker will immediately liquidate some or all of the investor's positions.

A company enters into a short futures contract to sell 100000 pounds of frozen orange juice for \$1.4 cents per pound. The initial margin is 30% and the maintenance margin is 20%.

The annual effective rate of interest is 4.5%. The account is settled every week.

What is the minimum next week price which would lead to a margin call?

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**Solution:** The initial balance in the margin account is  $(0.30) \times (100000) \times (1.4) = 42000$ . The minimum balance in the margin account is  $(0.20) \times (100000) \times (1.4) = 28000$ . After settlement next week balance is

$$42000(1.045)^{1/52} + 100000(1.4 - S_{1/52}).$$

A margin call happens if

$$28000 > 42000(1.045)^{1/52} + 100000(1.4 - S_{1/52}),$$

or

$$S_{1/52} > 1.4 - \frac{28000 - 42000(1.045)^{1/52}}{100000} = 1.540355672.$$

# Advantages of futures versus forwards

The two main advantages of futures versus forwards are liquidity and counter-party risk.

It is much easier to cancel before expiration a future contract than a forward contract.

Since the trade is made against a clearinghouse, a participant does face credit risk.

At the same time, the margin and the marking to market reduces the default risk.

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