[Solution Key]

KING SAUD UNIVERSITY
COLLEGE OF SCIENCES
DEPARTMENT OF MATHEMATICS

Semester 452 / MATH-244 (Linear Algebra) / Mid-term Exam 2

Max. Marks: 25 **Max. Time: $1\frac{1}{2}$ hr**

Question 1: [Marks: (1+2+2) + (1+2)]

a) **Determine** whether the following statements are true. **Justify** your answers.

i) $W = \{(a, b, c) \mid a, b, c$ are non-negative real numbers$\}$ is a subspace of $\mathbb{R}^3$.

Not true. For $-1 \in \mathbb{R}$ and $(1,0,0) \in W$, but $-1(1,0,0) = (-1,0,0) \notin W$. So W is not closed under scalar multiplication.

ii) For any fixed matrix $Y \in M_n(\mathbb{R})$, $\{A \in M_n(\mathbb{R}) \mid AY = YA\}$ is a subspace of the vector space $M_n(\mathbb{R})$ of all real matrices of type $n \times n$. W

True. (1) $0 \in W$ since $0Y = 0 = Y0$.
(2) If $A, B \in W$ then $AY = YA$ and $BY = YB \Rightarrow (A+B)Y = AY + BY = YA + YB = Y(A+B)$
$\Rightarrow A+B \in W$.
(3) If $s \in \mathbb{R}$ and $A \in W$ then $AY = YA \Rightarrow (sA)Y = s(AY) = sYA = Y(sA) \Rightarrow sA \in W$.

iii) Any **set of five** 2×2 matrices must be linearly dependent.

True. Since $\dim(M_{2\times 2}) = 4 \Rightarrow$ Any subset with more than 4 vectors is L.D.

b) **Consider** the vector subspace $W = \{p(x) \in P_2 \mid p(1) = p(2)\}$ of P_2, where P_2 denotes the vector space of all real polynomials in x with degree at most 2 under the usual addition and scalar multiplication. Then:

i) **Show** that $P_2 - W \neq \emptyset$.

$p(x) = x \in P_2$. Since $p(1) = 1 \neq p(2) = 2$, then $p(x) \notin W$. So $p(x) \in P_2 - W$.

ii) **Show** that $\{1, (x-1)(x-2)\}$ is a linearly independent subset and **explain why** it must be a basis for W. B

Since neither vector is a scalar multiple of the other they are L.I.
Since $1|_{x=1} = 1 = 1|_{x=2}$ and $(x-1)(x-2)|_{x=1} = 0 = (x-1)(x-2)|_{x=2} \Rightarrow 1, (x-1)(x-2) \in W$.
$\Rightarrow \dim W \geq 2$. From i) $\dim W < \dim P_2 = 3 \Rightarrow \dim W = 2$

$\Rightarrow B$ is a basis since B has 2 vectors and L.I.

Question 2: [Marks: 3 + 2 + 2 + 2]

Let $B := \{u_1 = (2,1),\ u_2 = (5,2)\}$ and $C := \{v_1 = (1,-2), v_2 = (-3,7)\}$. Then:

i) **Show** that both B and C are bases for the vector space $\mathbb{R}^2$.

B and C each has 2 vectors and neither of which is a multiple of the other and $\dim \mathbb{R}^2 = 2$
$\Rightarrow$ B and C are bases for $\mathbb{R}^2$.

ii) **Construct** the transition matrix ${}_CP_B$ from the basis B to the basis C.

$${}_CP_B = \left[[u_1]_C \quad [u_2]_C \right].$$

Solving $c_1(1,-2) + c_2(-3,7) = (2,1) \Rightarrow \begin{aligned} c_1 - 3c_2 &= 2 \quad \times 2 \\ -2c_1 + 7c_2 &= 1 \quad + \\ \hline c_2 &= 5 \end{aligned}$

$\Rightarrow c_1 = 17$

$$\Rightarrow [u_1]_C = \begin{bmatrix} 17 \\ 5 \end{bmatrix}$$

Solving $c_1(1,-2) + c_2(-3,7) = (5,2) \Rightarrow \begin{aligned} c_1 - 3c_2 &= 5 \quad \times 2 \\ -2c_1 + 7c_2 &= 2 \quad + \\ \hline c_2 &= 12 \end{aligned}$

$\Rightarrow c_1 = 41$

$$\Rightarrow [u_2]_C = \begin{bmatrix} 41 \\ 12 \end{bmatrix}$$

$$\therefore {}_CP_B = \begin{bmatrix} 17 & 41 \\ 5 & 12 \end{bmatrix}$$

iii) **Use** the matrix ${}_CP_B$ to **find** the transition matrix ${}_BP_C$.

$${}_BP_C = {}_CP_B^{-1} = \frac{1}{-1} \begin{bmatrix} 12 & -41 \\ -5 & 17 \end{bmatrix} = \begin{bmatrix} -12 & 41 \\ 5 & -17 \end{bmatrix}$$

iv) **Find** the coordinate vectors $[u_1]_C$ and $[v_2]_B$ by **using** the matrices ${}_CP_B$ and ${}_BP_C$, respectively.

$$[u_1]_C = \begin{bmatrix} 17 \\ 5 \end{bmatrix}, \quad [v_2]_B = \begin{bmatrix} 41 \\ -17 \end{bmatrix}$$

Question 3: [Marks: 2 + 3 + 3]

Let $RREF(A) = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ and $RREF(A^T) = \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ be the reduced row-echelon forms of a matrix A and its transpose A^T, respectively. Then, **find**:

i) $rank(A)$ and $nullity(A)$.

rank(A) = number of nonzero rows in $RREF(A) = 2$

Since rank(A) + nullity(A) = number of columns of $A = 3 \Rightarrow$ nullity$(A) = 3-2 = 1$.

ii) A basis for each of the vector spaces: $col(A)$, $row(A)$, $N(A)$.

The nonzero rows of $RREF(A^T)$ form a basis for col(A)

$\Rightarrow \{(1,0,1,2), (0,1,1,1)\}$ is a basis for Col(A)

The nonzero rows of $RREF(A)$ form a basis for row(A)

$\Rightarrow \{(1,0,1), (0,1,1)\}$ is a basis for row(A)

Since $0 \neq (1,1,-1) \in N(A)$ and $\dim(N(A)) = 1 \Rightarrow \{(1,1,-1)\}$ is a basis for $N(A)$.

iii) Three vectors u, v and w such that $u \in \mathbb{R}^3 \backslash row(A)$, $v \in \mathbb{R}^4 \backslash col(A)$ and $w \in \mathbb{R}^3 \backslash N(A)$.

In each case we need a vector not in the span of the corresponding basis:

For $N(A)$ take any nonscalar multiple of $(1,1,-1)$. For example $w = (1,0,0)$.

For row(A) and Col(A) we can use $RREF(A)$ and $RREF(A^T)$ and find a row to add so that the matrices stay in REF.

$$RREF(A) = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \qquad RREF(A^T) = \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Or $0 \; 0 \; 0 \; 1$

$\Rightarrow$ Take $u = (0,0,1)$ and $v = (0,0,1,0)$