[**Solution Key**]

KING SAUD UNIVERSITY
COLLEGE OF SCIENCES
DEPARTMENT OF MATHEMATICS

Semester 452 / MATH-244 (Linear Algebra) / Mid-term Exam 1

Max. Marks: 25 **Max. Time: $1\frac{1}{2}$ hr**

Question 1: [Marks: 3 + 4 + 3]

(a). Let $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & -2 & 1 \\ 2 & 1 & -2 \end{bmatrix}$. Compute A^2 and then use A^2 to find A^{-1}.

Solution: $A^2 = \begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix} = 9\mathbf{I}$ and so $A^{-1} = \frac{1}{9}A = \frac{1}{9}\begin{bmatrix} 1 & 2 & 2 \\ 2 & -2 & 1 \\ 2 & 1 & -2 \end{bmatrix}$. (Marks 1 + 2)

(b). Let $A = \begin{bmatrix} 1 & 1 & -1 & -2 \\ 1 & 1 & 0 & -2 \\ -1 & 0 & 1 & 1 \\ -2 & 0 & 0 & 1 \end{bmatrix}$. Find A^{-1} and then use A^{-1} to find $adj(A)$.

Solution: $[A|I] = \left[\begin{array}{cccc|cccc} 1 & 1 & -1 & -2 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & -2 & 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ -2 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$ 1 $\sim \dots \sim \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & -1 & 1 & -1 \\ 0 & 1 & 0 & 0 & 3 & -2 & 3 & -1 \\ 0 & 0 & 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & -2 & 2 & -1 \end{array}\right]$ and so $A^{-1} = \begin{bmatrix} 1 & -1 & 1 & -1 \\ 3 & -2 & 3 & -1 \\ -1 & 1 & 0 & 0 \\ 2 & -2 & 2 & -1 \end{bmatrix}$. ((Marks 1.5)

Next, $det(A) = 1$. Hence, $adj(A) = det(A)A^{-1} = \begin{bmatrix} 1 & -1 & 1 & -1 \\ 3 & -2 & 3 & -1 \\ -1 & 1 & 0 & 0 \\ 2 & -2 & 2 & -1 \end{bmatrix}$. (Marks 1 + 1.5)

(c). Let A be a 3×3 matrix with $det(A) = 2$. Evaluate $det(adj(A))$.

Solution: $det(adj(A)) = det(det(A)A^{-1}) = (det(A))^3(\det(A))^{-1} = 4$. (Marks 1 + 1.5 + 0.5)

Question 2: [Marks: 4 + 4]

(a). Let $A = \begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix}$. Find all matrices $M = \begin{bmatrix} x & y \\ z & t \end{bmatrix}$ such that

2 **Solution:** $AM = MA$ implies $\begin{cases} y - z = 0 \\ x - 3z - t = 0 \\ x - 3y - t = 0. \end{cases}$ This system of linear equations has solution set (Mark 1.5)

$\{(3z + t, z, z, t) \mid t, z \in \mathbb{R}\}$. Hence, $M = \begin{bmatrix} 3z + t & z \\ z & t \end{bmatrix}$ for all $t, z \in \mathbb{R}$. (Marks 2 + 0.5)

(b). Find the values of a, b and c so that (1,−2,3) is the solution of following system of linear equations:

$$\begin{aligned} ax + 2by + cz &= 6 \\ ax + 6by + cz &= -2 \\ 3ax + 4by + cz &= -8. \end{aligned}$$

3 **Solution:** Since (1,−2,3) is the solution of the above given system, we get the following system of linear equations:

$$\begin{aligned} a - 4b + 3c &= 6 \\ a - 12b + 3c &= -2 \\ 3a - 8b + 3c &= -8. \end{aligned}$$ (Marks 2)

Which has the unique solution $a = -5$, $b = 1$ and $c = 5$. (Marks 2)

Question 3: [Marks: 3 + 4]

(a). Use the Gauss-Jordan elimination method to solve the linear system $AX = B$, where:

$$A = \begin{bmatrix} 2 & -1 & -4 & 3 \\ 3 & -2 & -5 & 4 \\ 3 & -3 & -2 & 0 \end{bmatrix},\ X = \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

4 **Solution:** $\left[\begin{array}{cccc|c} 2 & -1 & -4 & 3 & 1 \\ 3 & -2 & -5 & 4 & 1 \\ 3 & -3 & -2 & 0 & 1 \end{array}\right] \sim \dots \sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & -7 & 4 \\ 0 & 1 & 0 & -5 & 3 \\ 0 & 0 & 1 & -3 & 1 \end{array}\right]$ (RREF). (Marks 1.5)

Hence, solution set of the given linear system is {(4+7t, 3+5t, 1+3t, t) | t ∈ ℝ}. (Marks 1.5)

(b). Find all the non-trivial solutions of the following homogeneous system:

$$\begin{aligned} 2x + 2y + 4z &= 0 \\ w - y - 3z &= 0 \\ 2w + 3x + y + z &= 0 \\ -2w + x + 3y - 2z &= 0. \end{aligned}$$

5 **Solution:** $\left[\begin{array}{cccc|c} 0 & 2 & 2 & 4 & 0 \\ 1 & 0 & -1 & -3 & 0 \\ 2 & 3 & 1 & 1 & 0 \\ -2 & 1 & 3 & -2 & 0 \end{array}\right] \sim \dots \sim \left[\begin{array}{cccc|c} 1 & 0 & -1 & -3 & 0 \\ 0 & 1 & 1 & -8 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$. (Marks 1.5)

Hence, the set of non-trivial solutions of the given linear system is {(t, - t, t, 0) | 0 ≠ t ∈ ℝ}. (Marks 2 + 0.5)

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1.

$$\left[\begin{array}{cccc|cccc} 1 & 1 & -1 & -2 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & -2 & 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ -2 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{A^{(-1)}_{1,2},\ A^{(1)}_{1,3},\ A^{(2)}_{1,4}} \left[\begin{array}{cccc|cccc} 1 & 1 & -1 & -2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 & 1 & 0 \\ 0 & 2 & -2 & -3 & 2 & 0 & 0 & 1 \end{array}\right] \xrightarrow{A^{(-2)}_{3,4}\ I_{2,3}} \left[\begin{array}{cccc|cccc} 1 & 1 & -1 & -2 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -2 & -1 & 0 & 0 & -2 & 1 \end{array}\right]$$

$$\xrightarrow{A^{(2)}_{3,4}} \left[\begin{array}{cccc|cccc} 1 & 1 & -1 & -2 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & -2 & 2 & -2 & 1 \end{array}\right] \xrightarrow{M^{(-1)}_{4}} \left[\begin{array}{cccc|cccc} 1 & 1 & -1 & -2 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & -2 & 2 & -1 \end{array}\right] \xrightarrow{A^{(1)}_{4,2},\ A^{(2)}_{4,1}} \left[\begin{array}{cccc|cccc} 1 & 1 & -1 & 0 & 5 & -4 & 4 & -2 \\ 0 & 1 & 0 & 0 & 3 & -2 & 3 & -1 \\ 0 & 0 & 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & -2 & 2 & -1 \end{array}\right]$$

$$\xrightarrow{A^{(1)}_{3,1}} \left[\begin{array}{cccc|cccc} 1 & 1 & 0 & 0 & 4 & -3 & 4 & -2 \\ 0 & 1 & 0 & 0 & 3 & -2 & 3 & -1 \\ 0 & 0 & 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & -2 & 2 & -1 \end{array}\right] \xrightarrow{A^{(-1)}_{2,1}} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & -1 & 1 & -1 \\ 0 & 1 & 0 & 0 & 3 & -2 & 3 & -1 \\ 0 & 0 & 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & -2 & 2 & -1 \end{array}\right]$$

Note only $I_{2,3}$ and $M_4^{(-1)}$ change det. by (-1) so they cancel each other $\Rightarrow |A| = 1$.

2.

$$\begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} x & y \\ z & t \end{bmatrix} = \begin{bmatrix} x & y \\ z & t \end{bmatrix} \cdot \begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 2x+z & 2y+t \\ x-z & y-t \end{bmatrix} = \begin{bmatrix} 2x+y & x-y \\ 2z+t & z-t \end{bmatrix}$$

$\Rightarrow z = y \ ,\ 2y+t = x-y \ ,\ x-z = 2z+t \ ,\ y-t = z-t$

$\Rightarrow y = z \ ,\ x = 3z+t \ ,\ x = 3z+t$

$\equiv \ y = z \ ,\ x = 3z+t \ ,\ z,t \in \mathbb{R}$

3

$$\left[\begin{array}{ccc|c} 1 & -4 & 3 & 6 \\ 1 & -12 & 3 & -2 \\ 3 & -8 & 3 & -8 \end{array}\right] \xrightarrow{A_{1,2}^{(-1)},\, A_{1,3}^{-3}} \left[\begin{array}{ccc|c} 1 & -4 & 3 & 6 \\ 0 & -8 & 0 & -8 \\ 0 & 4 & -6 & -26 \end{array}\right] \xrightarrow{M_2^{(-\frac{1}{8})}} \left[\begin{array}{ccc|c} 1 & -4 & 3 & 6 \\ 0 & 1 & 0 & 1 \\ 0 & 4 & -6 & -26 \end{array}\right]$$

$$\xrightarrow{A_{2,3}^{(-4)}} \left[\begin{array}{ccc|c} 1 & -4 & 3 & 6 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & -6 & -30 \end{array}\right] \xrightarrow{M_3^{(-\frac{1}{6})}} \left[\begin{array}{ccc|c} 1 & -4 & 3 & 6 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 5 \end{array}\right]$$

$$\Rightarrow c = 5, \quad b = 1, \quad a = 6 - 15 + 4 = -5$$

4

$$\left[\begin{array}{cccc|c} 2 & -1 & -4 & 3 & 1 \\ 3 & -2 & -5 & 4 & 1 \\ 3 & -3 & -2 & 0 & 1 \end{array}\right] \xrightarrow{A_{1,2}^{(-1)}\ I_{1,2}} \left[\begin{array}{cccc|c} 1 & -1 & -1 & 1 & 0 \\ 2 & -1 & -4 & 3 & 1 \\ 3 & -3 & -2 & 0 & 1 \end{array}\right] \xrightarrow{A_{1,2}^{(-2)},\, A_{1,3}^{(-3)}} \left[\begin{array}{cccc|c} 1 & -1 & -1 & 1 & 0 \\ 0 & 1 & -2 & 1 & 1 \\ 0 & 0 & 1 & -3 & 1 \end{array}\right]$$

$$\xrightarrow{A_{3,2}^{(2)},\, A_{3,1}^{(1)}} \left[\begin{array}{cccc|c} 1 & -1 & 0 & -2 & 1 \\ 0 & 1 & 0 & -5 & 3 \\ 0 & 0 & 1 & -3 & 1 \end{array}\right] \xrightarrow{A_{2,1}^{(1)}} \left[\begin{array}{cccc|c} 1 & 0 & 0 & -7 & 4 \\ 0 & 1 & 0 & -5 & 3 \\ 0 & 0 & 1 & -3 & 1 \end{array}\right]$$

$z = t, \quad y = 3t+1, \quad x = 5t+3, \quad w = 7t+4$

for any $t \in \mathbb{R}$.

$$S = \{(7t+4, 5t+3, 3t+1, t) : t \in \mathbb{R}\}$$

5

$$\left[\begin{array}{cccc|c} 0 & 2 & 2 & 4 & 0 \\ 1 & 0 & -1 & -3 & 0 \\ 2 & 3 & 1 & 1 & 0 \\ -2 & 1 & 3 & -2 & 0 \end{array}\right] \xrightarrow{M_1^{\frac{1}{2}},\, I_{12}} \left[\begin{array}{cccc|c} 1 & 0 & -1 & -3 & 0 \\ 0 & 1 & 1 & 2 & 0 \\ 2 & 3 & 1 & 1 & 0 \\ -2 & 1 & 3 & -2 & 0 \end{array}\right] \xrightarrow{A_{1,3}^{(-2)},\, A_{1,4}^{(2)}} \left[\begin{array}{cccc|c} 1 & 0 & -1 & -3 & 0 \\ 0 & 1 & 1 & 2 & 0 \\ 0 & 3 & 3 & 7 & 0 \\ 0 & 1 & 1 & -8 & 0 \end{array}\right]$$

$$\xrightarrow{A_{2,3}^{(-3)},\, A_{2,4}^{(-1)}} \left[\begin{array}{cccc|c} 1 & 0 & -1 & -3 & 0 \\ 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -10 & 0 \end{array}\right] \xrightarrow{A_{3,4}^{(10)}} \left[\begin{array}{cccc|c} 1 & 0 & -1 & -3 & 0 \\ 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{A_{3,2}^{(-2)},\, A_{3,1}^{(3)}} \left[\begin{array}{cccc|c} 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$\Rightarrow y=t,\ z=0,\ x=-t,\ w=t$ for $t \in \mathbb{R}$.

The non-trivial solutions are:

$$\{(t, -t, t, 0) : t \in \mathbb{R},\ t \neq 0\}$$