Quiz 2 نموذج حل

Name: **Student ID:**

Q1:Consider the following PDE:

$$u_{xx} - u_{yy} = 0 \longrightarrow (1)$$

1. Classify the equation.
2. Reduce the equation to its canonical form.
3. Solve the canonical form that you obtained in Part 2.

(1) $A = 1$, $B = 0$, $C = -1$

$\Delta = B^2 - 4AC = 0^2 - 4(1)(-1) = 4 > 0$

$\therefore$ the equation is hyper-bolic

(2) $Am^2 + Bm + C = 0$

$\Rightarrow m^2 - 1 = 0 \Rightarrow m^2 = 1 \Rightarrow \boxed{m = \pm 1}$

$\frac{dy}{dx} = -1 \quad \wedge \quad \frac{dy}{dx} = +1$

$dy = -dx \quad \wedge \quad dy = dx$

$\Rightarrow y + x = c_1 \quad \wedge \quad y - x = c_2$

$\therefore \xi = y + x \quad \wedge \quad \eta = y - x$

$\xi_x = 1$, $\xi_y = 1$, $\xi_{xx} = \xi_{yy} = \xi_{xy} = 0$

$\eta_x = -1$, $\eta_y = 1$, $\eta_{xx} = \eta_{yy} = \eta_{xy} = 0$

$U_x = U_\xi \xi_x + U_\eta \eta_x = U_\xi - U_\eta$

$U_{xx} = U_{\xi\xi} \xi_x + U_{\xi\eta} \eta_x - U_{\eta\xi} \xi_x - U_{\eta\eta} \eta_x$

$= U_{\xi\xi} - U_{\xi\eta} - U_{\xi\eta} + U_{\eta\eta} = \boxed{U_{\xi\xi} - 2U_{\xi\eta} + U_{\eta\eta}}$

$$U_y = U_\xi \xi_y + U_\eta \eta_y$$

$$U_y = U_\xi + U_\eta$$

$$U_{yy} = U_{\xi\xi}\xi_y + U_{\xi\eta}\eta_y + U_{\eta\xi}\xi_y + U_{\eta\eta}\eta_y$$

$$\boxed{U_{yy} = U_{\xi\xi} + 2U_{\xi\eta} + U_{\eta\eta}}$$

substitute in ①

$$\cancel{U_{\xi\xi}} - 2U_{\xi\eta} + \cancel{U_{\eta\eta}} - \cancel{U_{\xi\xi}} - 2U_{\xi\eta} - \cancel{U_{\eta\eta}} = 0$$

$$-4U_{\xi\eta} = 0$$

$$\boxed{U_{\xi\eta} = 0}$$

③ Integrate w.r.t η.

$$U_\xi = f(\xi)$$

integrate w.r.t ξ.

$$U(\xi,\eta) = F(\xi) + g(\eta)$$

$$\boxed{U(x,y) = F(y+x) + g(y-x)}$$

check:

$U_x = F' - g'$ $U_y = F' + g'$

$U_{xx} = F'' + g''$ $U_{yy} = F'' + g''$

$U_{xx} - U_{yy} = F'' + g'' - F'' + g'' = 0 = R.H.S$ ✓

نموذج حل

Quiz 3

Name: ______________________ **student ID:** ______________

Question I:

(a) Classify the following second order PDE in three variables $u_{xx} + 2u_{yy} + u_{zz} - u_{yx} - u_{zy} = u_{xy} + u_{yz}$.

$$N = \begin{vmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{vmatrix}$$

$$|\lambda I - N| = \begin{vmatrix} \lambda-1 & 1 & 0 \\ 1 & \lambda-2 & 1 \\ 0 & 1 & \lambda-1 \end{vmatrix} = (\lambda-1)\begin{vmatrix} \lambda-2 & 1 \\ 1 & \lambda-1 \end{vmatrix} - \begin{vmatrix} 1 & 1 \\ 0 & \lambda-1 \end{vmatrix} = (\lambda-1)[(\lambda-1)(\lambda-2)-1] - (\lambda-1)$$

$$= (\lambda-1)[(\lambda-2)(\lambda-1) - 1 - 1] = (\lambda-1)(\lambda^2 - \lambda - 2\lambda + \cancel{2} - \cancel{2}) = (\lambda-1)(\lambda)(\lambda-3) = 0$$

$\Rightarrow \boxed{\lambda=0}, \boxed{\lambda=1}, \boxed{\lambda=3} \Rightarrow \therefore$ the equation is parabolic

(b) Solve $2u_{xx} + 5u_{xy} + 2u_{yy} = 0$, where $u(0,t) = 1, u_x(0,t) = t$.

$$\Rightarrow u_{xx} + \frac{5}{2}u_{xy} + u_{yy} = 0$$

$$L = \frac{\partial^2}{\partial x^2} + \frac{5}{2}\frac{\partial}{\partial x}\frac{\partial}{\partial y} + \frac{\partial^2}{\partial y^2} = \left(\frac{\partial}{\partial x} + \frac{1}{2}\frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial x} + 2\frac{\partial}{\partial y}\right)$$

$a_1 = 1 \quad b_1 = \frac{1}{2} \quad c_1 = 0 \quad \& \quad a_2 = 1, \; b_2 = 2, \; c_2 = 0$

$$\therefore u(x,y) = f\left(\tfrac{1}{2}x - y\right) + g(2x - y) \quad (*)$$

$$u_x(x,y) = \tfrac{1}{2}f' + 2g'$$

apply the conditions:

- $u(0,t) = 1$

$\Rightarrow f(-t) + g(-t) = 1 \rightarrow (1)$

- $u_x(0,t) = t$

$\frac{1}{2}f'(-t) + 2g'(-t) = t \rightarrow (2)$

Derertiate (1) w.r.t to t

$\times 2 \quad -2f'(-t) - 2g'(-t) = 0 \rightarrow (3)$

add (2) & (3) $\quad -\frac{3}{2}f'(-t) = t \Rightarrow f'(-t) = -\frac{2}{3}t$

integrate w.r.t to t

$$\frac{f(-t)}{-1} = -\frac{\cancel{2}}{3}\frac{t^2}{\cancel{2}} + C$$

$$f(-t) = \frac{1}{3}t^2 + C$$

$w = -t \Rightarrow t = -w \quad f(w) = \frac{1}{3}w^2 + C$

substitute in (1)

$$g(-t) = 1 - \frac{1}{3}t^2 - C$$

$$g(w) = 1 - \frac{1}{3}w^2 - C$$

substitute in (*)

$$u(x,y) = \frac{1}{3}\left(\frac{1}{2}x - y\right)^2 + \cancel{C} + 1 - \frac{1}{3}(2x-y)^2 - \cancel{C}$$

$$= \frac{1}{3}\left(\frac{1}{4}x^2 - xy + y^2\right) + 1 - \frac{1}{3}(4x^2 - 4xy + y^2)$$

$$= \frac{1}{12}x^2 - \frac{1}{3}xy + \cancel{\frac{1}{3}y^2} + 1 - \frac{4}{3}x^2 + \frac{4}{3}xy - \cancel{\frac{1}{3}y^2}$$

$$\boxed{= -\frac{15}{12}x^2 + xy + 1}$$

$$xU_{xx} + U_x = 9x^2y^3 \qquad x \neq 0,\ y \neq 0$$

$$\frac{\partial}{\partial x}(u_x x) = 9x^2y^3$$

integrate w.r.t x $\Rightarrow$

$$U_x x = 3x^3y^3 + f(y)$$

$$U_x = 3x^2y^3 + \frac{1}{x} f(y)$$

integrate w.r.t x:

$$\boxed{U(x,y) = x^3y^3 + \ln(x)\cdot f(y) + g(y)}$$