نموذج حل

King Saud University جامعة الملك سعود

College of Science.
Department of Mathematics

كلية العلوم
قسم الرياضيات

First Midterm Exam
Academic Year 1446 Hijri- First Semester

Exam Information معلومات الامتحان			
Course name	Introduction to Partial Differential Equations		اسم المقرر
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Classroom No.	G14		رقم قاعة الاختبار
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Section No.		رقم الشعبة
Serial Number		الرقم التسلسلي

General Instructions:

- Your Exam consists of 7 PAGES (except this paper)
- Keep your mobile and smart watch out of the classroom.
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تعليمات عامة:

- عدد صفحات الامتحان ٧ صفحة. (بإستثناء هذه الورقة)
- يجب إبقاء الهواتف والساعات الذكية خارج قاعة الامتحان.
-

هذا الجزء خاص بأستاذ المادة

This section is ONLY for instructor

#	Course Learning Outcomes (CLOs)	Related Question (s)	Points	Final Score
1	C.L.O 2-1(2 marks)	QII(b)		
2	C.L.O 2-2 (4+2 marks)	QI QII(a)		
3	C.L.O 2-3 (6+3+3 marks)	QIII QIV		
4				
5				
6				
7				
8				

EXAM COVER PAGE

Question I: Obtain the general solution of the given equation:

$$u_{xx} - 2u_{xy} - 3u_{yy} = e^{2x+3y} + \sin(2x+y) \rightarrow (*)$$

$$u_{xx} - 2u_{xy} - 3u_{yy} = 0$$

$$\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial x} - 3\frac{\partial}{\partial y}\right) = 0$$

$a_1 = 1 \quad b_1 = 1 \quad , c_1 = 0$

$a_2 = 1 \quad b_2 = -3 \quad c_2 = 0$

$\therefore U(x,y) = f(x-y) + g(-3x-y)$

$\Rightarrow \boxed{U(x,y) = f(x-y) + g(3x+y)}$

• $U_p = A e^{2x+3y} + B\sin(2x+y) + C\cos(2x+y)$

$(U_p)_x = 2A e^{2x+3y} + 2B\cos(2x+y) - 2C\sin(2x+y)$

$(U_p)_{xy} = 6A e^{2x+3y} - 2B\sin(2x+y) - 2C\cos(2x+y)$

$(U_p)_{xx} = 4A e^{2x+3y} - 4B\sin(2x+y) - 4C\cos(2x+y)$

$(U_p)_y = 3A e^{2x+3y} + B\cos(2x+y) - C\sin(2x+y)$

$(U_p)_{yy} = 9A e^{2x+3y} - B\sin(2x+y) - C\cos(2x+y)$

Substitute in $(*)$:

$$4A e^{2x+3y} \cancel{- 4B\sin(2x+y)} \cancel{- 4C\cos(2x+y)} - 12A e^{2x+3y} \cancel{+ 4B\sin(2x+y)}$$
$$\cancel{+ 4C\cos(2x+y)} - 27A e^{2x+3y} + 3B\sin(2x+y) + 3C\cos(2x+y) =$$
$$e^{2x+3y} + \sin(2x+y)$$

Comparing $\therefore -35A = 1 \Rightarrow \boxed{A = \frac{-1}{35}}$, $3B = 1 \Rightarrow \boxed{B = \frac{1}{3}}$, $3C = 0 \Rightarrow \boxed{C = 0}$

$\therefore$

$$\boxed{U(x,y) = f(x-y) + g(3x+y) - \frac{1}{35} e^{2x+3y} + \frac{1}{3}\sin(2x+y)}$$

Question II:

(a) Solve the PDE $xyu_{xy} = 1$. $\Rightarrow u_{xy} = \frac{1}{xy}$

$\Rightarrow$ integrate w.r.t y $\Rightarrow U_x = \frac{1}{x} \ln y + f(x)$

$\Rightarrow$ integrate w.r.t x

$U = \ln x \ln y + F(x) + g(y)$, when $F'(x) = f(x)$

(b) Determine the condition under which the function

$$u(x, y) = \cos(ax - by)e^{(ax-by)}$$

is a harmonic. is a solution of Laplace equation : $U_{xx} + U_{yy} = 0$

$U_x = -a\sin(ax-by)\, e^{ax-by} + a\cos(ax-by)\, e^{ax-by}$

$U_{xx} = \cancel{-a^2\cos(ax-by)\, e^{ax-by}} - a^2\sin(ax-by)\, e^{ax-by} - a^2\sin(ax-by) \cdot e^{ax-by}$
$\quad + \cancel{a^2\cos(ax-by)\, e^{ax-by}}$

$\boxed{U_{xx} = -2a^2\sin(ax-by)\, e^{ax-by}} \to (1)$

$U_y = b\sin(ax-by)\, e^{(ax-by)} - b\cos(ax-by)\, e^{ax-by}$

$U_{yy} = \cancel{-b^2\cos(ax-by)\, e^{ax-by}} - b^2\sin(ax-by)\, e^{ax-by} - b^2\sin(ax-by) \cdot e^{ax-by}$
$\quad + \cancel{b^2\cos(ax-by)\, e^{ax-by}}$

$\boxed{U_{yy} = -2b^2\sin(ax-by)\, e^{ax-by}} \to (2)$

add (1) & (2) $\quad -2(a^2+b^2)\sin(ax-by)\, e^{ax-by} = 0$

$\therefore$ the condition is:

$\boxed{a^2+b^2 \neq 0}$ or $\boxed{a^2 \neq -b^2}$

Question III: Solve the Cauchy problem $yu_{xy} + 2u_x = x^2, y \neq 0,$ in $\mathbb{R}^2$, that satisfies:

$$u(s,2s)_{|\Gamma_0} = \frac{s^3}{6}, \qquad u_n(s,2s)_{|\Gamma_0} = -\frac{s^2}{\sqrt{5}}$$

where

$\Gamma_0 = \{(x,y)|x = s, y = 2s, s \in \mathbb{R}^*\}$, and u_n is a derivative of u in the direction of Γ_0.

$$u_{xy} + \frac{2}{y}u_x = \frac{x^2}{y}$$

integrate w.r.t x :-

$$u_y + \frac{2}{y}u = \frac{x^3}{3y} + f(y)$$

integrate w.r.t y (by using integration factor $e^{\int \frac{2}{y}dy} = e^{2\ln y} \Rightarrow e^{\ln y^2} = \underline{y^2}$

$$y^2 u_y + 2yu = \frac{1}{3}x^3 y + f(y)\, y^2$$

$$\Rightarrow \frac{\partial}{\partial y}(y^2 u) = \frac{1}{3}x^3 y + f(y)\, y^2$$

integrate w.r.t y

$$\Rightarrow y^2 u = \frac{1}{3}x^3 \frac{y^2}{2} + F(y) + g(x), \text{ where } F'(y) = y^2 f(y)$$

$$\Rightarrow \boxed{u(x,y) = \frac{1}{6}x^3 + \frac{F(y)}{y^2} + \frac{g(x)}{y^2}} \longrightarrow (*)$$

we need u_x & u_y to apply the conditions:

$$\frac{y^2 F'(y) - g(x)(2y)}{y^4}$$

$$u_x = \frac{1}{2}x^2 + \frac{g'(x)}{y^2}$$

$$u_y = \frac{y^2 F'(y) - F(y)(2y)}{y^4} + g(x)\left(\frac{-2}{y^3}\right)$$

$$u_y = \frac{F'(y)}{y^2} - \frac{2}{y^3}F(y) - \frac{2g(x)}{y^3}$$

$$\left.\begin{aligned} \alpha = s &\Rightarrow \alpha' = 1 \\ \beta = 2s &\Rightarrow \beta' = 2 \end{aligned}\right\} \sqrt{\alpha'^2 + \beta'^2} = \sqrt{5}$$

$$u_n = \frac{\alpha' u_y - \beta' u_x}{\sqrt{\alpha'^2 + \beta'^2}} = \frac{1}{\sqrt{5}}\left[\frac{F'(y)}{y^2} - \frac{2}{y^3}F(y) - \frac{2g(x)}{y^3} - x^2 - \frac{2g'(x)}{y^2}\right]$$

• $U(s,2s) = \frac{s^3}{6}$

$$\Rightarrow \cancel{\frac{s^3}{6}} = \frac{1}{6}\cancel{s^3} + \frac{1}{\cancel{4s^2}}F(2s) + \frac{g(s)}{\cancel{4s^2}} \Rightarrow \boxed{F(2s) + g(s) = 0} \to ①$$

• $u_n(s,2s) = \frac{-s^2}{\sqrt{5}}$

$$\Rightarrow \frac{-s^2}{\sqrt{5}} = \frac{1}{\sqrt{5}}\left[\frac{1}{4s^2}F'(2s) - \frac{2}{8s^3}F(2s) - \frac{2g(s)}{8s^3} - s^2 - \frac{2g'(s)}{4s^2}\right]$$

$$\xRightarrow{\times 4s^2} \cancel{-4s^4} = F'(2s) - \frac{\cancel{2}}{\cancel{2}s}F(2s) - \frac{\cancel{2}}{\cancel{2}s}g(s) \cancel{-4s^4} - 2g'(s)$$

$$\Rightarrow F'(2s) - \frac{1}{s}F(2s) - \frac{1}{s}g(s) - 2g'(s) = 0$$

$$\Rightarrow \boxed{F'(2s) - \frac{1}{s}\left[F(2s) + g(s)\right] - 2g'(s) = 0} \to \boxed{2}$$

Substituate ① in ②

$$F'(2s) - 2g'(s) = 0 \to \boxed{3}$$

Derviative ① w.r.t s:

$$2F'(2s) + g'(s) = 0 \xRightarrow{\times 2} 4F'(2s) + 2g'(s) = 0 \to \boxed{4}$$

add $\boxed{3}$ & ④

$$\Rightarrow 5F'(2s) = 0 \Rightarrow F'(2s) = 0 \xRightarrow{\text{integrate w.r.t s}} \frac{F(2s)}{2} = C$$

$$\boxed{F(2s) = 2C}$$

substituate in ①

$$\boxed{g(s) = -2C}$$

Substituate in (*)

$$U(x,y) = \frac{1}{6}x^3 + \frac{2\cancel{C}}{y^2} - \frac{2\cancel{C}}{y^2}$$

$$\Rightarrow \boxed{u(x,y) = \frac{1}{6}x^3}$$

check $u_x = \frac{1}{2}x^2$ $u_{xy} = 0$ $y(0) + 2\cdot\frac{1}{2}x^2 = x^2$

$x^2 = x^2$ ✓

Question IV:

(a) Consider the series solution

$$u(r,\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(\frac{r}{a}\right)^n (a_n \cos n\theta + b_n \sin n\theta),$$

for Dirichlet problem for circle of radius a, if the boundary condition is $u(a,\theta) = \sin\theta,\ \ 0 \le \theta \le 2\pi$. Obtain $u(r,\theta)$?

$$U(a,\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \underbrace{\left(\frac{a}{a}\right)^n}_{1} (a_n \cos n\theta + b_n \sin n\theta)$$

$$\Rightarrow \sin\theta = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\theta + b_n \sin n\theta$$

We can use Feurior serise or directly Comparing:

$$\sin\theta = \frac{a_0}{2} + a_1 \cos\theta + b_1 \sin\theta + a_2 \cos 2\theta + b_2 \sin 2\theta + \cdots$$

$$\frac{a_0}{2} = 0 \Rightarrow \boxed{a_0 = 0}$$

$$\boxed{a_n = 0} \quad \forall n$$

$$b_1 = 1 \quad \boxed{b_n = 0} \quad \forall n > 1$$

$$\therefore\ b_n = \begin{cases} 1 & n = 1 \\ 0 & n \neq 1 \end{cases}$$

$$\therefore\ U(r,\theta) = 0 + \left(\frac{r}{a}\right)^1 (0 \cdot \cos\theta + 1 \cdot \sin\theta) + 0$$

$$\boxed{U(r,\theta) = \frac{r}{a} \sin\theta}$$

(b) Consider the Laplace equation in spherical coordinate

$$u_{rr} + \frac{2}{r}u_r + \frac{1}{r^2 sin^2\phi}u_{\theta\theta} + \frac{1}{r^2}u_{\phi\phi} + \frac{\cot\phi}{r^2}u_\phi = 0.$$

If u is a symmetry around z axis, then the solution is given by

$$u(r,\theta) = \sum_{n=0}^{\infty}(a_n r^n + b_n r^{-n-1})P_n(\cos\phi).$$

Find the solution at the boundary condition

$u(1,\phi) = f(\phi), 0 \le r < 1$ *where*

$$f(\phi) = \begin{cases} 110 \text{ if } 0 \le \phi < \frac{\pi}{2}. \\ 0 \text{ if } \frac{\pi}{2} < \phi \le \pi. \end{cases}$$

(Hint: $\|P_n\|^2 = \frac{2}{2n+1}$, $P_0(x) = 1, P_1(x) = x, P_2(x) = \frac{1}{2}(3x^2 - 1), P_3(x) = \frac{1}{2}(5x^3 - 3x)$.)

u is a symmetry around z: so,

$$U_{rr} + \frac{2}{r}U_r + \frac{1}{r^2}u_{\phi\phi} + \frac{\cot\phi}{r^2}U_\phi = 0$$

since $0 \le r < 1 \Rightarrow b_n = 0$

$$\Rightarrow U(r,\theta) = \sum_{n=0}^{\infty} a_n r^n P_n(\cos\phi)$$

apply the condition: $u(1,\phi) = f(\phi)$

$$\Rightarrow \sum_{n=0}^{\infty} a_n (1)^n P_n(\cos\phi) = \begin{cases} 110 & 0 \le \phi \le \frac{\pi}{2} \\ 0 & \frac{\pi}{2} < \phi \le \pi \end{cases}$$

to find $a_n = \frac{1}{\|P_n\|^2}\int_0^\pi (f(\phi) \cdot P_n(\cos\phi))\, d\phi$

$$= \frac{1}{\|P_n\|^2}\left[\int_0^{\frac{\pi}{2}} 110 \cdot P_n(\cos\phi)\, d\cos\phi + \int_{\frac{\pi}{2}}^{\pi} 0 \cdot P_n(\cos\phi)\, d\cos\phi\right]$$

$$= \frac{110}{\|P_n\|^2}\int_0^{\frac{\pi}{2}} P_n(\cos\phi)\sin\phi\, d\phi$$

$$= \frac{11\cancel{0}(2n+1)}{\cancel{2}}\int_0^{\frac{\pi}{2}} P_n(\cos\phi)\sin\phi\, d\phi$$

$$a_n = 55(2n+1)\int_0^{\frac{\pi}{2}} P_n(\cos\phi)\sin\phi\, d\phi$$

let $x = \cos\phi \Rightarrow dx = -\sin\phi\, d\phi$

$\phi = 0 \Rightarrow x = 1$

$\phi = \frac{\pi}{2} \Rightarrow x = 0$

$$a_n = 55(2n+1)\int_0^1 P_n(x)\, dx$$

$$a_0 = 55\int_0^1 P_0\, dx = 55(x)\Big|_0^1 = \boxed{55}$$

$$a_1 = 55(3)\int_0^1 P_1\, dx = 165\int_0^1 x\, dx = 165\frac{x^2}{2}\Big|_0^1 = 165\left(\frac{1}{2}\right) = \boxed{\frac{165}{2}}$$

$$a_2 = 55(5)\int_0^1 P_2\, dx = 275\int_0^1 \frac{1}{2}(3x^2-1)\, dx = \frac{275}{2}\left(x^3 - x\Big|_0^1\right) = \boxed{0}$$

$$a_3 = 55(7)\int_0^1 P_3\, dx = 385\int_0^1 \frac{1}{2}(5x^3-3x)\, dx = \frac{385}{2}\left(\frac{5}{4}x^4 - \frac{3}{2}x^2\right)\Big|_0^1$$

$$= \frac{385}{2}\left[\frac{5}{4} - \frac{3}{2}\right] = \frac{385}{2}\left[\frac{5}{4} - \frac{6}{4}\right] = \frac{385}{2}\left(\frac{-1}{4}\right) = \boxed{-\frac{385}{8}}$$

$$u(r,\phi) = a_0 P_0(\cos\phi) + a_1 r P_1(\cos\phi) + a_2 r^2 P_2(\cos\phi) + a_3 r^3 P_3(\cos\phi) + \cdots$$

$$\boxed{u(r,\phi) = 55 P_0(\cos\phi) + \frac{165}{2} r P_1(\cos\phi) - \frac{385}{8} r^3 P_3(\cos\phi) + \cdots}$$

Good Luck

حل آخر للتمرين

Question III: Solve the Cauchy problem $yu_{xy} + 2u_x = x^2, y \neq 0$, in $\mathbb{R}^2$, that satisfies:

$$u(s,2s)_{|\Gamma_0} = \frac{s^3}{6}, \qquad u_n(s,2s)_{|\Gamma_0} = -\frac{s^2}{\sqrt{5}}$$

where
$\Gamma_0 = \{(x,y) | x = s, y = 2s, s \in \mathrm{R}^*\}$, and u_n is a derivative of u in the direction of Γ_0.

$$y u_{xy} + 2u_x = x^2$$

let $v = u_x$

$$\Rightarrow y v_y + 2v = x^2$$

$$\overset{\div y}{\Rightarrow} v_y + \frac{2}{y} v = \frac{x^2}{y}$$

integrate w.r.t y (By using integration factor) $e^{\int \frac{2}{y} dy} = e^{\ln y^2} = y^2$

$$\Rightarrow y^2 v_y + 2yv = x^2 y$$

$$\Rightarrow \frac{\partial}{\partial y}(v y^2) = x^2 y$$

$$\Rightarrow v y^2 = x^2 \frac{y^2}{2} + f(x)$$

$$\Rightarrow v = \frac{1}{2}x^2 + \frac{f(x)}{y^2} \Rightarrow u_x = x^2 + \frac{f(x)}{y^2}$$

integrate w.r.t x:

$$\boxed{u(x,y) = \frac{1}{6}x^3 + y^{-2}F(x) + g(y)} \; (*), \text{ where } F'(x) = f(x)$$

$$u_x = \frac{1}{2}x^2 + y^{-2}F'(x)$$

$$u_y = -2y^{-3}F(x) + g'(y)$$

$\alpha = s \Rightarrow \alpha' = 1$ $\quad \sqrt{\alpha'^2 + \beta'^2} = \sqrt{5}$

$\beta = 2s \Rightarrow \beta' = 2$

$$U_n = \frac{\alpha' u_y - \beta' u_x}{\sqrt{\alpha'^2 + \beta'^2}} = \frac{1}{\sqrt{5}}\left[\frac{-2}{y^3}F(x) + g'(y) - x^2 - \frac{2F'(x)}{y^2}\right]$$

apply the conditions:

- $u(s, 2s) = \frac{s^3}{6}$

$$\Rightarrow \frac{s^3}{6} = \cancel{\frac{1}{6}(s)^3} + \frac{F(s)}{4s^2} + g(2s) \Rightarrow \boxed{F(s) + 4s^2 g(2s) = 0} \rightarrow 1$$

- $u_n(s, 2s) = -\frac{s^2}{\sqrt{5}}$

$$\cancel{\frac{s^2}{\sqrt{5}}} = \cancel{\frac{1}{\sqrt{5}}}\left[\frac{-2}{8s^3}F(s) + g'(2s) - \cancel{s^2} - \frac{2F'(s)}{4s^2}\right]$$

$$\Rightarrow \boxed{\frac{-1}{4s^3}F(s) + g'(2s) + \frac{F'(s)}{2s^2} = 0} \rightarrow 2$$

from 1 : $\frac{F(s)}{4s^2} + g'(2s) = 0$

$$\Rightarrow \frac{4s^2F'(s) - F(s)(8s)}{16s^4} + 2g'(2s) = 0$$

$$\Rightarrow \frac{1}{4s^2}F'(s) - \frac{F(s)}{8s^3} + 2g'(2s) = 0$$

$$\Rightarrow g'(2s) = \frac{F(s)}{16s^3} - \frac{F'(s)}{8s^2}$$

substituate in 2

$$\frac{-1 \times 4}{4s^3}F(s) + \frac{F(s)}{16s^3} - \frac{F'(s)}{8s^2} + \frac{4 \times F'(s)}{4 \times 2s^2} = 0$$

$16s^3 \times$ $\quad \frac{3F(s)}{16s^3} + \frac{3F'(s)}{8s^2} = 0 \Rightarrow \frac{F(s)}{16s^3} + \frac{F'(s)}{8s^2} = 0$

$$F(s) + 2sF'(s) = 0$$

$$F'(s) + \frac{1}{2s}F(s) = 0 \qquad \left(e^{\int \frac{1}{2s}ds} = e^{\frac{1}{2}\ln s} = s^{\frac{1}{2}}\right)$$

$$s^{\frac{1}{2}}F'(s) + \frac{1}{2}s^{-\frac{1}{2}}F(s) = 0$$

$$\frac{\partial}{\partial s}\left(s^{\frac{1}{2}}F(s)\right) = 0 \qquad s^{\frac{1}{2}}F(s) = C$$

$$F(s) = Cs^{-\frac{1}{2}}$$

$$\boxed{F(s) = C s^{-\frac{1}{2}}} \longrightarrow A$$

substitual in 1

$$C s^{-\frac{1}{2}} + 4 s^2 g(2s) = 0$$

$$\frac{\cancel{4s^2}\, g(2s)}{\cancel{4s^2}} = \frac{-C s^{-\frac{1}{2}}}{4s^2}$$

$-\frac{1}{2} - \frac{4}{2} = -\frac{3}{2}$

$$g(2s) = -\frac{1}{4} C s^{-\frac{3}{2}}$$

$$\boxed{g(w) = -\frac{1}{4} C \left(\frac{w}{2}\right)^{-\frac{3}{2}}} \longrightarrow B$$

substitute in $*$

$$u(x,y) = \frac{1}{6}x^3 + y^{-2}\left(C x^{-\frac{1}{2}}\right) + \left(-\frac{1}{4} C \left(\frac{y}{2}\right)^{-\frac{3}{2}}\right)$$

$$\frac{1}{\sqrt{2^3}}$$

$2 \cdot 2^2$

$= 2\sqrt{2}$

$$\boxed{u(x,y) = \frac{1}{6}x^3 + \frac{C}{\sqrt{x}\, y^2} - \frac{C}{8\sqrt{2}}\, y^{-\frac{3}{2}}}$$

check

$$U_x = \frac{1}{2}x^2 + \frac{C}{y^2}\left(-\frac{1}{2} x^{-\frac{3}{2}}\right)$$

$$U_y = \frac{C}{\sqrt{x}}\left(-2y^{-3}\right) - \frac{C}{8\sqrt{2}}\left(-\frac{3}{2} y^{-\frac{5}{2}}\right)$$

$$U_{xy} = -\frac{C}{2} x^{-\frac{3}{2}} \left(-2y^{-3}\right)$$

$$yU_{xy} + 2U_x = x^2$$

$$+C x^{-\frac{3}{2}} \cancel{y^{-2}} + x^2 - \frac{C}{\cancel{y^2}} x^{-\frac{3}{2}} = x^2$$

✓ check ✓