نموذج حل

College of Science.
Department of Mathematics

King Saud University جامعة الملك سعود
كلية العلوم
قسم الرياضيات

First Midterm Exam
Academic Year 1446 Hijri- First Semester

Exam Information معلومات الامتحان			
Course name	Introduction to Partial Differential Equations		اسم المقرر
Course Code	425 Math		رمز المقرر
Exam Date	2024-10-14	1446-04-11	تاريخ الامتحان
Exam Time	10: 00 AM		وقت الامتحان
Exam Duration	2 hours	ساعتان	مدة الامتحان
Classroom No.	G14		رقم قاعة الاختبار
Instructor Name	د. هدى الرشيدي		اسم استاذ المقرر

Student Information معلومات الطالب		
Student's Name		اسم الطالب
ID number		الرقم الجامعي
Section No.		رقم الشعبة
Serial Number		الرقم التسلسلي

General Instructions:

- Your Exam consists of 6 PAGES (except this paper)
- Keep your mobile and smart watch out of the classroom.
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تعليمات عامة:

- عدد صفحات الامتحان 6 صفحة. (بإستثناء هذه الورقة)
- يجب إبقاء الهواتف والساعات الذكية خارج قاعة الامتحان.
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هذا الجزء خاص بأستاذ المادة

This section is ONLY for instructor

#	Course Learning Outcomes (CLOs)	Related Question (s)	Points	Final Score
1	C.L.O 1-1 (3 marks)	QI(1(a)) QI(2) QIV(1)	3	20/20
2	C.L.O 2-1 (6 marks)	QI(1(b,c)) QI(3) QIV(2)	6	
3	C.L.O 2-2 (6 marks)	QI(1(d,e)) QII QIV(3)	6	
4	C.L.O 2-3 (5 marks)	QI(1(f)) QIII	5	
5				
6				
7				
8				

EXAM COVER PAGE

King Saud University
Department of Mathematics
1st Semester 1446 H

425–MATH
Midterm 1
Duration: 2 hours

Question I:[3 points]

1. Choose the correct answer:

(a) The equation $u_{xx} = x^2 u_{yy}$ is

i. Hyperbolic for all x.
ii. Hyperbolic for $x > 0$.
iii. Hyperbolic for $x \neq 0$.
iv. None of previous.

(b) A partial differential equation of the family surfaces of $z = x^2 + y^2$ is

i. $yz_y - xz_x = 0$.
ii. $yz_x - xz_y = 0$.
iii. $yz_x + xz_y = 0$.
iv. None of previous.

(c) The subsidiary equation of $x^2 u_x + u - 1 = \frac{y-1}{3} u_y$ is given by

i. $\frac{dx}{x^2} = \frac{3dy}{1-y} = \frac{du}{1-u}$.
ii. $\frac{dx}{x^2} = \frac{3dy}{y-1} = \frac{du}{1-u}$.
iii. $\frac{dx}{x^2} = \frac{3dy}{1-y} = \frac{du}{u-1}$.
iv. None of previous.

(d) The general solution of the partial differential equation $5\frac{\partial u}{\partial x} + 4\frac{\partial u}{\partial y} + u = 0$ equals

i. $u(x,y) = f(4x-5y)e^{-\frac{x}{5}}$.
ii. $u(x,y) = f(4x-5y)e^{\frac{x}{5}}$.
iii. $u(x,y) = f(4x-5y)$.
iv. None of previous.

(e) A particular solution of PDE $3u_x + 3u_y = x^3$ is

i. $\frac{x^4}{4}$.
ii. $\frac{x^4}{3}$.
iii. $\frac{x^4}{12}$.
iv. None of previous.

(f) The Cauchy problem $u_x + u_y = u$ with initial condition $x_0 = t$, $y_0 = t$, $u_0 = sint$, has

i. One solution.
ii. No solution.
iii. Infinitely many solutions.

2. Classify each of the following PDEs as linear, quasilinear, or nonlinear and state its order and homogeneity:[2 points]

(a) $\frac{\partial^2 z}{\partial x^2} = \left(1 + \frac{\partial z}{\partial y}\right)^{\frac{1}{2}}$.

non-linear

Second-order

quasi linear.

non-homogeneous

(b) $u_y u_{xxy}^2 + u_x u_{yy} - u_z^2 = (z + xy^2)u$.

third-order non-linear

homogenous. non quasi linear

3. Prove that the PDE which has integral surface $F(v,w) = 0$ where $v = x + y + u$ and $w = x^2 + y^2 - u^2$ such that $u = u(x,y)$, can be written as [2 points]

$$(y+u)u_x - (x+u)u_y = x - y.$$

let $w = x^2 + y^2 - u^2$ $\quad$ $v = x + y + u$

$f < \begin{matrix} v < x, y, u \\ w < x, y, u \end{matrix}$

$$f_v(v_x + v_u u_x) + f_w(w_x + w_u u_x) = 0$$

$$f_v(v_y + v_u u_y) + f_w(w_y + w_u u_y) = 0$$

$$\Rightarrow f_v(1 + u_x) + f_w(2x - 2uu_x) = 0$$

$$f_v(1 + u_y) + f_w(2y - 2uu_y) = 0$$

$$\frac{f_v(1+u_x)}{f_w} + (2x - 2uu_x) = 0 \Rightarrow \frac{f_v}{f_w} = \frac{-(2x - 2uu_x)}{1 + u_x}$$

also $\frac{f_v}{f_w} = \frac{-(2y - 2uu_y)}{1 + u_y}$

$$\Rightarrow \frac{(2y - 2uu_y)}{1 + u_y} = \frac{2x - 2uu_x}{1 + u_x}$$

$$\Rightarrow 2y - 2uu_y + (2y - \cancel{2uu_y})u_x = 2x - 2uu_x + (2x - \cancel{2uu_x})u_y$$

$$2y - 2uu_y - 2xu_y + 2yu_x - 2uu_x - 2x = 0$$

$$(y - u)u_x - (x + u)u_y = x - y.$$

Question II:
Find the integral surface of $x^2u_x - xyu_y + u = 0$.[3 points]

$P(\cancel{xy}) = x^2 \quad Q\,\cancel{xy} = -xy \quad R = -u$

$$\frac{dx}{x^2} = \frac{dy}{-xy} = \frac{du}{-u}$$

① $\frac{dx}{x^2} = \frac{dy}{-xy} \Rightarrow \frac{dx}{x} = \frac{dy}{-y}$

$\Rightarrow \ln x + \ln y = \ln c_1$

$\ln xy = \ln c_1$

$\boxed{xy = c_1}$

② $\frac{dx}{x^2} = \frac{du}{-u} \Rightarrow x^{-2}dx = -\frac{du}{u}$

$\Rightarrow \frac{-1}{x} = -\ln u + \ln c_2 \Rightarrow \frac{-1}{x} = \ln \frac{c_2}{u} \Rightarrow \frac{c_2}{u} = e^{-\frac{1}{x}}$

$\Rightarrow \boxed{ue^{-\frac{1}{x}} = c_2}$

$\therefore F(xy, ue^{-\frac{1}{x}}) = 0$

or $ue^{-\frac{1}{x}} = f(xy)$

$\boxed{u = e^{\frac{1}{x}} f(xy)}$

Question III:

Find the explicit solution of the following partial differential equation

$$xu_x + yu_y = x^2 - y$$

which passes through the curve $u(1,y) = y$.[4.5 points]

$$\frac{dx}{x} = \frac{dy}{y} = \frac{du}{x^2-y}$$

① $\frac{dx}{x} = \frac{dy}{y} \Rightarrow \ln x = \ln y + \ln c_1 \Rightarrow x = yc_1$

$$\Rightarrow \boxed{\frac{x}{y} = c_1}$$

② $\frac{dy}{y} = \frac{du}{x^2-y} \Rightarrow \frac{dy}{y} \times \frac{du}{y^2c_1^2-y}$

$\frac{\cancel{y}(yc_1^2-1)}{\cancel{y}} dy = du \Rightarrow (c_1^2 y - 1)dy = du$

$\Rightarrow c_1^2 \frac{y^2}{2} - y = u + c_2$

$c_1^2 y^2 - 2y = 2u + 2c_2$

$\frac{x^2}{y^2} y^2 - 2y - 2u = c_3$ $(c_3 = 2c_2)$

$\boxed{2u + 2y - x^2 = c_4}$ $, c_4 = -c_3$

$\Rightarrow F\left(\frac{x}{y}, 2u + 2y - x^2\right) = 0$

$2u + 2y - x^2 = f\left(\frac{x}{y}\right)$

apply the condition $u(1,y) = y$ let $y = t$

$2t + 2t - 1 = f\left(\frac{1}{t}\right) \Rightarrow f\left(\frac{1}{t}\right) = 4t - 1 \Rightarrow f(w) = \frac{4}{w} - 1$

$\therefore 2u + 2y - x^2 = \frac{4}{\frac{x}{y}} - 1$

$2u = \frac{4y}{x} - 1 + x^2 - 2y$ 4 $\Rightarrow \boxed{u = \frac{4y}{2x} - 2 + \frac{x^2}{2} - y}$

Question IV:

1. Show that the one-dimensional wave equation

$$u_{tt} - c^2 u_{xx} = 0$$

is hyperbolic?[0.5 point]

2. Find an equivalent canonical form.[3 points]
3. Obtain the general solution.[2 points]

① $A=1,\ B=0\quad C=-c^2$

$\Delta = B^2-4AC = 0-4(1)(-c^2) = 4c^2$

if $c\neq 0$ hyperpolic

② $m_{1,2} = \dfrac{-0 \pm \sqrt{4c^2}}{2} \Rightarrow m_1 = \dfrac{2c}{2},\ m_2 = \dfrac{-2c}{2}$

$\boxed{m_1 = c},\ \boxed{m_2=-c}$

$\dfrac{dx}{dt} = -c \qquad \dfrac{dx}{dt} = c$

$dx = -c\,dt \qquad dx = c\,dt$

$x = -ct + a_1 \qquad x = ct + b$

$\boxed{x+ct = a_1} \qquad \boxed{x-ct = b}$

let - $\xi = x+ct \qquad \eta = x-ct$

$\xi_x = 1\quad \xi_t = c\quad \xi_{xt} = 0 = \xi_{xx} = \xi_{tt} = 0$

$\eta_x = 1\quad \eta_t = -c\quad \eta_{xt} = 0 = \eta_{xx} = \eta_{tt}$

$U_t = U_\xi \xi_t + U_\eta \eta_t = cU_\xi - cU_\eta$

$U_{tt} = c(U_{\xi\xi}\xi_t + U_{\xi\eta}\eta_t) - c(U_{\eta\xi}\xi_t + U_{\eta\eta}\eta_t)$

$U_{tt} = c^2U_{\xi\xi} - c^2U_{\xi\eta} - c^2U_{\xi\eta} + c^2U_{\eta\eta}$

$U_{tt} = c^2U_{\xi\xi} - 2c^2U_{\xi\eta} + c^2U_{\eta\eta}$

$$U_x = U_\xi \xi_x + U_\zeta \zeta_x$$

$$U_x = U_\xi + U_\zeta$$

$$u_{xx} = U_{\xi\xi}\xi_x + U_{\xi\zeta}\zeta_x + U_{\zeta\xi}\xi_x + U_{\zeta\zeta}\zeta_x$$

$$U_{xx} = U_{\xi\xi} + 2U_{\xi\zeta} + U_{\zeta\zeta}$$

Substituate in $U_{tt} - c^2 u_{xx} = 0$

$$\cancel{c^2 U_{\xi\xi}} - 2c^2 U_{\xi\zeta} + \cancel{c^2 U_{\zeta\zeta}} - \cancel{c^2 U_{\xi\xi}} - 2c^2 U_{\xi\zeta} - \cancel{c^2 U_{\zeta\zeta}} = 0$$

$$-4c^2 U_{\xi\zeta} = 0$$

$\Rightarrow \quad U_{\xi\zeta} = 0 \qquad$ since $c \neq 0$

3 $\quad U_{\xi\zeta} = 0$

integrate w.r.t ζ

$$U_\xi = f(\xi)$$

integrate w.r.t ξ

$$U = F(\xi) + g(\zeta)$$

$$\boxed{u = F(x+ct) + g(x-ct)}$$

Good Luck