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QI: (1)

ⓐ $3u_{tt} + 10u_{xt} + 3u_{xx} = \sin(x+t) \rightarrow (*)$

① Solve $3u_{tt} + 10u_{xt} + 3u_{xx} = 0$

$$L = 3\frac{\partial^2}{\partial t^2} + 10\frac{\partial}{\partial t}\frac{\partial}{\partial x} + 3\frac{\partial^2}{\partial x^2}$$

$$= \left(3\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} + 3\frac{\partial}{\partial x}\right) \Rightarrow \quad \begin{matrix} a_1 = 3 & b_1 = 1 & c_1 = c_2 = 0 \\ a_2 = 1 & b_2 = 3 & \end{matrix}$$

$u(x,t) = f(t-3x) + g(3t-x)$, where f & g are arbitirary functions.

② Solve (*) , suppose the particular solution is given by

$$u_p = A\sin(x+t) + B\cos(x+t)$$

$(U_p)_x = A\cos(x+t) - B\sin(x+t)$

$(U_p)_{xx} = -A\sin(x+t) - B\cos(x+t)$

$(U_p)_t = A\cos(x+t) - B\sin(x+t)$

$(U_p)_{tt} = -A\sin(x+t) - B\cos(x+t)$

$(U_p)_{xt} = -A\sin(x+t) - B\cos(x+t)$

substituate in (*)

$$-3A\sin(x+t) - 3B\cos(x+t) + 10A\sin(x+t) - 10B\cos(x+t)$$
$$+ 3A\sin(x+t) - 3B\cos(x+t) = \sin(x+t)$$

$-3A - 10A - 3A = 1 \Rightarrow A = \frac{1}{-16}$

$-3B - 10B - 3B = 0 \Rightarrow \boxed{B = 0}$

$$\therefore \; U_p = -\frac{1}{16}\sin(x+t)$$

$$\therefore \; u(x,t) = f(t-3x) + g(3t-x) - \frac{1}{16}\sin(x+t)$$

QI ① ⓑ $z_{yy} + x z_y = x^2 \quad , x \neq 0$

Direct integration.

Integrate w.r.t y :

$$\Rightarrow \quad z_y + x z = x^2 y + f(x)$$

integration factors $e^{\int x dy} = e^{xy}$

$$z_y \cdot e^{xy} + x z e^{xy} = x^2 y e^{xy} + e^{xy} f(x)$$

$$\frac{\partial}{\partial y}\left(z e^{xy}\right) = x^2 \cdot y e^{xy} + e^{xy} F(x)$$

integrat w.r.t y

$$z e^{xy} = x^2 \int y e^{xy} dy + \int e^{xy} f(x)\, dy$$

$$z e^{xy} = x^2\left(\frac{y}{x} e^{xy} - \frac{1}{x^2} e^{xy}\right) + \frac{e^{xy}}{x} f(x) + g(x)$$

y	e^{xy}
1	$\frac{e^{xy}}{x}$
0	$\frac{e^{xy}}{x^2}$

$$z = \left(xy e^{xy} - e^{xy} + \frac{e^{xy}}{x} f(x) + g(x)\right) e^{-xy}$$

$$\boxed{z = xy - 1 + \frac{1}{x} f(x) + e^{-xy} g(x)}$$

QI ② $u_{xy} + u_x = x$

integrate w.r.t x.

$$u_y + u = \frac{x^2}{2} + g(y)$$

integration factor $e^{\int dy} = e^y$

$$e^y u_y + u e^y = \frac{x^2}{2} e^y + g(y) e^y$$

$$\frac{\partial}{\partial y}\left(e^y u\right) = \frac{1}{2} x^2 e^y + g(y) e^y$$

$$e^y u = \frac{1}{2} x^2 e^y + G(y) \qquad \text{when } G'(y) = g(y) e^y$$

$$u = \frac{1}{2} x^2 + e^{-y} G(y)$$

$x = t , y = t$

$u = t^2 \quad u_y = 0$

$u(t,t) = t^2 \quad t^2 = \frac{1}{2}t^2 + e^{-t}G(t)$

$$\frac{1}{2}t^2 e^t = G(t)$$

$$u_y(x,y) = -e^{-y}G(y) + e^{-y}G'(y)$$

$u_y(t,t) = 0$

$$0 = -\cancel{e^{-t}}G(t) + \cancel{e^{-t}}G'(t)$$

$$G(t) = G'(t)$$

$$\therefore \quad u(x,y) = \frac{1}{2}x^2 + e^{-y}\cdot\frac{1}{2}y^2 e^y$$

$$\boxed{u(x,y) = \frac{1}{2}\left(x^2 + y^2\right)}$$

Q II $\quad u_{xx} + 4u_{xy} + u_x = 0$

(1) $\quad A = 1, \; B = 4, \; C = 0$

$\Phi = B^2 - 4AC = 4^2 = 16 > 0$

$\therefore$ the equation is hyperbolic.

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(2) $\quad m_{1,2} = \frac{-4 \pm \sqrt{16}}{2}, \quad m_1 = \frac{-4+4}{2} = 0$

$$m_2 = \frac{-4-4}{2} = -\frac{8}{2} = -4$$

$\frac{dy}{dx} = 0 \qquad \frac{dy}{dx} = 4$

$dy = 0 \qquad dy = 4\,dx$

$\boxed{y = C_1} \qquad y = 4x + C_2$

$\boxed{y - 4x = C_2}$

$\xi = y \qquad \eta = y - 4x$

$$\xi_x = \xi_{xx} = \xi_{xy} = 0 \qquad \xi_y = 1 \qquad \xi_{yy} = 0$$

$$\zeta_x = -4 \qquad \zeta_y = 1 \qquad , \zeta_{xx} = \zeta_{yy} = \zeta_{xy} = 0$$

$$\begin{tikzcd}[row sep=small] & & x \\ & \xi \arrow[ur] \arrow[r] & y \\ u \arrow[ur] \arrow[dr] & & x \\ & \zeta \arrow[ur] \arrow[r] & y \end{tikzcd}$$

$$U_x = U_\xi \xi_x + U_\zeta \zeta_x$$

$$U_x = 0 + (-4)\, U_\zeta = -4U_\zeta$$

$$U_{xx} = -4\left(U_{\zeta\xi}\xi_x + U_{\zeta\zeta}\zeta_x\right)$$

$$U_{xx} = -4\left(0 - 4U_{\zeta\zeta}\right) = 16\, U_{\zeta\zeta}$$

$$U_{xy} = -4\left(U_{\zeta\xi}\xi_y + U_{\zeta\zeta}\zeta_y\right)$$

$$= -4\left(U_{\zeta\xi} + U_{\zeta\zeta}\right)$$

Substituate in (*)

$$16\, U_{\zeta\zeta} + 4\left(-4\left(U_{\zeta\xi} + U_{\zeta\zeta}\right) + (-4U_\zeta)\right) = 0$$

$$\cancel{16U_{\zeta\zeta}} - 16U_{\zeta\xi} \cancel{-16U_{\zeta\zeta}} - 4U_\zeta = 0$$

$$-16\, U_{\zeta\xi} - 4U_\zeta = 0$$

$$\cancel{16U} \quad \boxed{4U_{\zeta\xi} + U_\zeta = 0}$$

(3) integrate w.r.t ζ:

$$4U_\xi + U = f(\xi)$$

$$U_\xi + \frac{1}{4}U = \frac{1}{4}f(\xi)$$

integration factor $e^{\int \frac{1}{4} d\xi} = e^{\frac{1}{4}\xi}$

$$U_\xi e^{\frac{1}{4}\xi} + \frac{1}{4}e^{\frac{1}{4}\xi}U = \frac{1}{4}f(\xi)\, e^{\frac{1}{4}\xi}$$

$$\frac{\partial}{\partial \xi}\left(U e^{\frac{1}{4}\xi}\right) = \frac{1}{4} f(\xi)\, e^{\frac{1}{4}\xi}$$

$$U e^{\frac{1}{4}\xi} = \frac{1}{4} G(\xi) + g(\eta), \quad G'(\xi) = f(\xi)\, e^{\frac{1}{4}\xi}$$

$$U = \frac{+1}{4} e^{-\frac{1}{4}\xi} G(\xi) + g(\eta)\, e^{-\frac{1}{4}\xi}$$

$$U(x,y) = \frac{1}{4} e^{-\frac{1}{4}y} G(y) + g(y-4x) \cdot e^{-\frac{1}{4}y}$$

$$\boxed{U(x,y) = \frac{1}{4} e^{-\frac{1}{4}y} G(y) + e^{-\frac{1}{4}y} g(y-4x)}$$

Q III

$$U_{xx} - 2U_{xy} - 3u_{yy} = 0$$

$$L = \left(\frac{\partial^2}{\partial x^2} - 2\frac{\partial}{\partial x}\frac{\partial}{\partial y} - 3\frac{\partial^2}{\partial y^2} \right)$$

$$= \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} \right)\left(\frac{\partial}{\partial x} - 3\frac{\partial}{\partial y} \right)$$

$a_1 = 1$, $b_1 = 1$ $c_1 = 0$, $a_2 = -3$, $b_2 = 1$ $c_2 = 0$

$$\boxed{u(x,y) = f(x-y) + g(x+3y)}$$

apply the condition $u(s,s) = \sin s$

$$\Rightarrow \boxed{\sin s = f(0) + g(4s)} \Rightarrow$$

To apply the condition $u_n(s,s) = 1$, we will find U_n first

$$U_n = \frac{\alpha' U_y - \beta' U_x}{\sqrt{\alpha'^2 + \beta'^2}}$$

$U_y = -f'(x-y) + 3g'(x+3y)$, $\alpha' = \beta' = 1 \Rightarrow \sqrt{\alpha'^2 + \beta'^2} = \sqrt{2}$

$U_x = f'(x-y) + g'(x+3y)$

$$\Rightarrow U_n = \frac{1}{\sqrt{2}}\left(-f' + 3g' - f' - g'\right) \Rightarrow \boxed{u_n = \frac{1}{\sqrt{2}}\left[-2f' + 2g'\right]}$$

$$\boxed{u_n(x,y) = \frac{2}{\sqrt{2}}\left[g'(x+3y) - f'(x-y) \right]}$$

$$u_n(s,s) = \frac{2}{\sqrt{2}}\left[g'(4s) - f'(0) \right] = 1$$

$$g'(4s) - f'(0) = \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{2}}{\sqrt{2}} = \frac{1}{\sqrt{2}}$$

So we have from two condition

$$f(0) + g(4s) = \sin s \longrightarrow (1)$$

$$f'(0) - g'(4s) = \frac{-1}{\sqrt{2}} \longrightarrow (2)$$

differeniate (1) w.r.t s $\Rightarrow$ $f'(0) + 4g'(4s) = \cos s \rightarrow (3)$

or

~~in (2) g(4s) is~~

integrate (2) $\quad f'(0)s - \frac{1}{4}g(4s) = \frac{-1}{\sqrt{2}}s \rightarrow (4)$

from (4) $g(4s)$ is plynomial of first order

~~while in (1) g(4s) is~~

and this is contradiction with (1)

Hence there is no solution.

QIV $\frac{V_{xx}}{V} = -\frac{W_{yy}}{W} = \lambda^2 \qquad \lambda \neq 0$

$\Rightarrow$ ① $V_{xx} - \lambda^2 V = 0$

$m^2 - \lambda^2 = 0 \Rightarrow m^2 = \lambda^2 \Rightarrow m = \pm\sqrt{\lambda^2}$

$m = \pm\lambda$

$\boxed{V(x) = C_1 \cosh\lambda x + C_2 \sinh \lambda x}$

② $\frac{W_{yy}}{W} + \lambda^2 = 0$

$W_{yy} + \lambda^2 W = 0$

$m^2 + \lambda^2 = 0$

$m^2 = -\lambda^2$

$m = \pm \lambda i$

$\boxed{W(y) = a \cos \lambda y + b \sin \lambda y}$

$u_x(x,y) = V_x = C_1 \lambda \sinh \lambda x + C_2 \lambda \cosh \lambda x$

$u_y(x,y) = W_y = -a\lambda \sin \lambda y + b\lambda \cos \lambda y$

apply the condition :

$u_x(0,y) = 0 \Rightarrow 0 = C_2 \lambda \Rightarrow \boxed{C_2 = 0}$

$u(x,0) = 0 \Rightarrow \boxed{0 = a}$

$u_y(x,1) = 0 \Rightarrow b\lambda \cos \lambda = 0 \Rightarrow b \neq 0, \lambda \neq 0 \Rightarrow \cos \lambda = 0$

$\Rightarrow \lambda = \left(n\pi - \frac{\pi}{2}\right) \qquad n \in \mathbb{N}$

$\lambda_n = (n - \frac{1}{2})\pi$

so

$u_n(x,y) = B_n \cosh \lambda_n x \cdot \sin \lambda_n y \qquad B_n = C_1 b$

$u(x,y) = \sum_{n=1}^{\infty} u_n(x,y) = \sum_{n=1}^{\infty} B_n \cosh \lambda_n x \cdot \sin \lambda_n y$

apply the Last condition: $u(1,y)=4$

$$4=\sum_{n=1}^{\infty} B_n \cosh \lambda_n \cdot \sin \lambda_n y \cdot$$

from the orthogonal function on $[0,1]$, we have

$$\int_0^1 4 \sin\left(\frac{2n-1}{2}\pi y\right) dy = B_n \cosh \overset{\left(\frac{2n-1}{2}\right)\pi}{\lambda_n} \int_0^1 \sin^2 \overset{\left(\frac{2n-1}{2}\right)\pi}{\lambda_n} y \, dy$$

$$\Rightarrow \frac{-4}{\frac{(2n-1)}{2}} \cos\left(\frac{2n-1}{2}\right)\pi y \Big|_0^1 = B_n \cosh\left(\frac{2n-1}{2}\right)\pi \int_0^1 \frac{1-\cos 2\frac{2n-1}{2}\pi y}{2} \, dy$$

$$\frac{-8}{2n-1}(0-1) = \frac{B_n}{2} \cosh\left(\frac{2n-1}{2}\right)\pi \left[y - \frac{\sin(2n-1)\pi y}{2n-1} \Big|_0^1 \right.$$

$$\frac{8}{2n-1} = \frac{1}{2} B_n \cosh \frac{2n-1}{2} \pi \underbrace{\left[(1-0) - \left(\frac{0-0}{2n-1}\right)\right]}_{=1}$$

$$\frac{8}{2n-1} = \frac{1}{2} B_n \cosh \frac{2n-1}{2} \pi$$

$$B_n = \frac{8}{2n-1} \cdot \frac{2}{\cosh\left(\frac{2n-1}{2}\right)\pi}$$

$$\boxed{B_n = \frac{16}{(2n-1)\cosh\left(\frac{2n-1}{2}\pi\right)}}$$

$$\therefore \boxed{u(x,y) = \sum_{n=1}^{\infty} \frac{16}{(2n-1)\cosh\left(\frac{2n-1}{2}\right)\pi} \cdot \cosh\left(\frac{2n-1}{2}\pi x\right) \sin\left(\frac{2n-1}{2}\pi y\right)}$$