Q I

(1) $\frac{y^2 z}{x} z_x + zx z_y - y^2 = 0$

$P = \frac{y^2 z}{x}$, $Q = xz$ $R = y^2$

$$\frac{dx}{\frac{y^2 z}{x}} = \frac{dy}{xz} = \frac{dz}{y^2}$$

(1) $\frac{x\,dx}{y^2 \cancel{z}} = \frac{dy}{x\cancel{z}} \Rightarrow x^2 dx = y^2 dy \Rightarrow y^2 dy - x^2 dx = 0$

$\frac{y^3}{3} - \frac{x^3}{3} = c_1 \Rightarrow y^3 - x^3 = c_2$ $3c_1 = c_2$

(2) $\frac{x\,d\cancel{x}}{\cancel{y^2} z} = \frac{dz}{\cancel{y^2}} \Rightarrow x\,dx = z\,dz \Rightarrow z\,dz - x\,dx = 0$

$\frac{z^2}{2} - \frac{x^2}{2} = c_3 \Rightarrow z^2 - x^2 = c_4$, $2c_3 = c_4$

The general solution is

$$F(y^3 - x^3, z^2 - x^2) = 0$$

or $z^2 - x^2 = f(y^3 - x^3)$

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(2) $3U_x - 2U_y + u = x \rightarrow (1)$

$A = 3$, $B = -2$

let $\xi = Bx - Ay = -2x - 3y \Rightarrow \xi = -2x - 3\eta$

$+2x = -3\eta - \xi$

$\eta = y$

$x = \frac{1}{2}(-3\eta - \xi)$

$U_x = U_\xi \cdot \xi_x + U_\eta \eta_x = -2U_\xi$

$U_y = U_\xi \xi_y + U_\eta \eta_y = -3U_\xi + U_\eta$

substitute in (1)

$$3(-2U_\xi) - 2(-3U_\xi + U_\eta) + U = \frac{1}{2}(-3\eta - \xi)$$

$$-6\cancel{U_\xi}+6\cancel{U_\xi}-2U_\eta+U=\frac{1}{2}(-3\eta-\xi)$$

$$-2U_\eta+U=\frac{-3}{2}\eta-\frac{1}{2}\xi$$

$$U_\eta+\frac{1}{-2}U=\frac{+3}{4}\eta+\frac{1}{4}\xi$$

integration factor :-

$$e^{\int-\frac{1}{2}d\eta}=e^{-\frac{1}{2}\eta}$$

$$e^{-\frac{1}{2}\eta}\cdot U_\eta-\frac{1}{2}e^{-\frac{1}{2}\eta}U=\left(\frac{3}{4}\eta+\frac{1}{4}\xi\right)e^{-\frac{1}{2}\eta}$$

$$\frac{\partial}{\partial\eta}\left(U e^{-\frac{1}{2}\eta}\right)=\frac{3}{4}\eta e^{-\frac{1}{2}\eta}+\frac{1}{4}e^{-\frac{1}{2}\eta}\xi$$

integrate w.r.t η

$$U e^{-\frac{1}{2}\eta}=\frac{1}{4}\xi\frac{e^{-\frac{1}{2}\eta}}{-\frac{1}{2}}+\int\frac{3}{4}\eta e^{-\frac{1}{2}\eta}d\eta$$

$z=\eta \qquad dv=e^{-\frac{1}{2}\eta}d\eta$

$dz=d\eta \qquad v=-2e^{-\frac{1}{2}\eta}$

$$U e^{-\frac{1}{2}\eta}=-\frac{1}{2}\xi e^{-\frac{1}{2}\eta}+\frac{3}{4}\left(-2\eta e^{-\frac{1}{2}\eta}+2\int e^{-\frac{1}{2}\eta}d\eta\right)$$

$$U e^{-\frac{1}{2}\eta}=-\frac{1}{2}\xi e^{-\frac{1}{2}\eta}-\frac{3}{2}\eta e^{-\frac{1}{2}\eta}+\frac{3}{2}\frac{e^{-\frac{1}{2}\eta}}{-\frac{1}{2}}+f(\xi)$$

$$U e^{-\frac{1}{2}\eta}=-\frac{1}{2}\xi e^{-\frac{1}{2}\eta}-\frac{3}{2}\eta e^{-\frac{1}{2}\eta}-3e^{-\frac{1}{2}\eta}+f(\xi)$$

$$U(\xi,\eta)=-\frac{1}{2}\xi-\frac{3}{2}\eta-3+f(\xi)e^{\frac{1}{2}\eta}$$

$$U(x,y) = -\frac{1}{2}(-2x-3y) - \frac{3}{2}y - 3 + f(-2x-3y)\,e^{\frac{1}{2}y}$$

check:

$$U_x = -\frac{1}{2}(-2) - 2f'e^{\frac{1}{2}y} = 1 - 2e^{\frac{1}{2}y}f'$$

$$U_y = -\frac{1}{2}(-3) - \frac{3}{2} + \frac{fe^{\frac{1}{2}y}}{2} + e^{\frac{1}{2}y}(-3)f'$$

$$= \cancel{\frac{3}{2}} - \cancel{\frac{3}{2}} + \frac{1}{2}e^{\frac{1}{2}y}f - 3f'e^{\frac{1}{2}y}$$

sub in (1)

$$LHS = 3(1 - 2e^{\frac{1}{2}y}f') - 2\left(\frac{1}{2}e^{\frac{1}{2}y}f - 3f'e^{\frac{1}{2}y}\right) +$$

$$-\frac{1}{2}(-2x-3y) - \frac{3}{2}y - 3 + fe^{\frac{1}{2}y}.$$

$$= \cancel{3} - \cancel{6e^{\frac{1}{2}y}f'} - \cancel{e^{\frac{1}{2}y}f} + \cancel{6f'e^{\frac{1}{2}y}} + x + \cancel{\frac{3}{2}y} - \cancel{\frac{3}{2}y} - \cancel{3}$$
$$\cancel{+fe^{\frac{1}{2}y}}$$

$$= x = R.H.S \checkmark$$

QI

B $\quad xu_x + yu_y = x^2 - y$

$$\frac{dx}{x} = \frac{dy}{y} = \frac{du}{x^2-y}$$

① $\frac{dx}{x} = \frac{dy}{y} \Rightarrow \ln x - \ln y = \ln c_1 \Rightarrow \boxed{\frac{x}{y} = c_1}$

$\Rightarrow \cancel{x = c_1 y \Rightarrow x^2 = c_1^2 y^2} \quad \cancel{x = c_1 y} \quad y = \frac{x}{c_1}$

② $\frac{d\cancel{x}}{\cancel{x}} = \frac{du}{x^2-y} = \frac{du}{x^2 - \frac{x}{c_1}}$

$$\frac{dx}{\cancel{x}} = \frac{du}{\cancel{x}(x-\frac{1}{c_1})} \Rightarrow (x - \frac{1}{c_1})\,dx = du$$

$\frac{x^2}{2} - \frac{1}{c_1}x = u + c_2 \Rightarrow \frac{x^2}{2} - \frac{x}{\frac{x}{y}} - u = c_2$

$\boxed{\frac{x^2}{2} - y - u = c_2}$

$$F\left(\frac{x}{y}, \frac{x^2}{2} - y - u\right) = 0$$

$\Rightarrow \frac{x^2}{2} - y - u = f\left(\frac{x}{y}\right)$

$x = 1, \; y = t, \; u = t$

$\frac{1}{2} - t - t = f\left(\frac{1}{t}\right)$

$\frac{1}{2} - 2t = f\left(\frac{1}{t}\right)$

let $\frac{1}{t} = \omega \Rightarrow t = \frac{1}{\omega} \Rightarrow \boxed{f(\omega) = \frac{1}{2} - 2\frac{1}{\omega}}$

$\frac{x^2}{2} - y - u = \frac{1}{2} - 2\frac{y}{x}$

$\boxed{\frac{x^2}{2} - y - u - \frac{1}{2} + 2\frac{y}{x} = 0}$

Q1 (c) $U_x + U_y = 1$

$$\frac{dx}{1} = \frac{dy}{1} = \frac{du}{1}$$

(1) $dx = dy \Rightarrow x - y = c_1$

(2) $dx = du \Rightarrow x - u = c_2$

$F(x-y, x-u) = 0$

or

$x - u = f(x-y)$

(1) $u(x,x) = x$, $x = t$, $y = t$, $u = t$

$t - t = f(0) \Rightarrow f(0) = 0$

infinity many solution $\quad \begin{vmatrix} P & Q \\ \alpha' & \beta' \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} = 0$

$$\frac{P}{\alpha'} = \frac{Q}{\beta'} = \frac{R}{\gamma'}$$

$$\frac{1}{1} = \frac{1}{1} = \frac{1}{1}$$

(2) $u(x,0) = h(x)$ $\quad x = t$, $y = 0$ $\quad u = h(t)$

$t - h(t) = f(t)$

$\Rightarrow x - u = x - y - h(x-y)$

one solution $\quad \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} = 0 - 1 = -1 \neq 0$

(3) $u(x,x) = 1$ $\quad x = t$, $y = t$, $u = 1$

$t - 1 = f(t-t) \Rightarrow f(0) = t - 1$ $\quad$ no solution

$\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} = 0 \quad \frac{R}{\gamma'} = \frac{1}{0} \neq \frac{P}{\alpha'}$

Q II

(A) $u(x,y) = e^{x} G(2x-y)$

$u_x = e^{x} G(2x-y) + 2e^{x} G'(2x-y)$

$u_y = -e^{x} G'(2x-y)$

$\Rightarrow G' = -u_y e^{-x}$ & $G(2x-y) = e^{-x} u$

$u_x = e^{x} \cdot e^{-x} u + 2e^{x}(-u_y e^{-x})$

$u_x = u - 2u_y$

$$\boxed{u_x + 2u_y - u = 0}$$

(B) $u_{tt} - 4u_{xx} = 0$ $\quad u(x,0) = e^{-x} = f(x)$

$u_t(x,0) = 0 = g(x)$

$$u(x,t) = \frac{1}{2}\left[f(x-2t) + f(x+2t)\right] + \frac{1}{2\cdot 2}\int_{x-2t}^{x+2t} g(s)\,ds$$

$$= \frac{1}{2}\left[e^{-(x-2t)} + e^{-(x+2t)}\right] + \frac{1}{2\cdot 2}\int_{x-2t}^{x+2t} 0\,ds$$

$$u(x,t) = \frac{1}{2} e^{-x+2t} + \frac{1}{2} e^{-x-2t} + \underline{\underline{A}}$$

Q III $\quad xU_{xx} + 2U_x = 0$

$y^2 = 4ax, \ u = 0 \quad \& \quad y^2 = -4ax, \ u = 1$

let $v = U_x \Rightarrow v_x = U_{xx}$

$xv_x + 2v = 0$

$v_x + \frac{2}{x} v = 0$

integrating factor $e^{\int \frac{2}{x} dx} = e^{2\ln x} = x^2$

$x^2 v_x + 2xv = 0$

$\frac{\partial}{\partial x}(x^2 v) = 0$

$x^2 v = G(y)$

$\boxed{v = G(y) x^{-2}}$

$u_x = G(y) x^{-2}$

integrate w.r.t x : $U(x,y) = G(y) \frac{x^{-1}}{-1} + F(y)$

$\therefore U(x,y) = -\frac{y^2}{8ax} + \frac{1}{2}$

$U(x,y) = -\frac{G(y)}{x} + F(y)$

apply $y^2 = 4ax, \ u = 0$

$0 = \frac{-G(y)}{\frac{y^2}{4a}} + F(y)$

$F(y) = \frac{4aG(y)}{y^2}$

$\Rightarrow F(y) = \frac{4a}{y^2} \cdot \frac{y^2}{8a} = \frac{1}{2}$

$y^2 = -4ax \quad U = 1$

$1 = \frac{-G(y)}{\frac{y^2}{-4a}} + F(y)$

$1 = \frac{4aG(y)}{y^2} + F(y)$

$1 = \frac{4a}{y^2} G(y) + \frac{4aG(y)}{y^2}$

$1 = \frac{8a}{y^2} G(y)$

$G(y) = \frac{y^2}{8a}$

Q (IV)

$$U(x,t) = V(x)\,W(t)$$

$$\frac{\cancel{V}\, W_t}{\cancel{V} W} = \frac{\cancel{V}_{xx}\,\cancel{W}}{V\,\cancel{W}}$$

$$\frac{V_{xx}}{V} = \frac{W_t}{W} = \alpha$$

① $V_{xx} - \alpha V = 0$

$m^2 - \alpha = 0$

$m^2 = \alpha$

$m = \pm\sqrt{\alpha}$

$\alpha = 0 \Rightarrow m = 0$

$\alpha > 0 \Rightarrow \sqrt{\alpha} > 0$

$\alpha < 0 \Rightarrow \sqrt{\alpha} = \sqrt{-\alpha} = \pm i\alpha$

$$V(x) = \begin{cases} a + bx & \alpha = 0 \\ a\cosh\sqrt{\alpha}\,x + b\sinh\sqrt{\alpha}\,x & \alpha > 0 \\ a\cos\sqrt{\alpha}\,x + b\sin\sqrt{\alpha}\,x & \alpha < 0 \end{cases}$$

apply the conditions:

$u(0,t) = U(a,t) = 0$

① $\alpha = 0$

$V(0) = 0$ & $V(a) = 0$

$\alpha = 0$ & $ba = 0 \Rightarrow b = 0$

$a = 0$

trivial solution

② $\alpha > 0$

$a\cosh\sqrt{\alpha}\,0 + b\sinh 0 = 0$

$a = 0$

& $b\sinh\sqrt{\alpha}\cdot a = 0 \Rightarrow b = 0$ trivial

② $W_t - \alpha W = 0$

$e^{-\alpha dt} = e^{-\alpha t}$

$e^{-\alpha t} W_t - \alpha e^{-\alpha t} W = 0$

$\frac{\partial}{\partial t}\left(e^{-\alpha t} W\right) = 0$

$\boxed{W = C_1 e^{+\alpha t}}$ $\alpha > 0$

$W = C_1 e^{-\alpha t}$ $\alpha < 0$

$W = C_1$ $\alpha = 0$

③ $\alpha < 0$

$a\cos 0 + b\sin 0 = 0$

$\Rightarrow a = 0$

& $b\sin\sqrt{\alpha}\cdot a = 0$

$b \neq 0 \Rightarrow \sin\sqrt{\alpha}\,a = 0$

$\Rightarrow \sqrt{\alpha}\,a = n\pi$ $n \in \mathbb{N}$

$\sqrt{\alpha} = \frac{n\pi}{a}$

$\alpha = \left(\frac{n\pi}{a}\right)^2$

$$U_n(x,t) = b_n \, e^{-\alpha_n t} \cdot \sin\sqrt{\alpha_n}\, x$$

The general solution

$$U(x,t) = \sum_{n=1}^{\infty} b_n \, e^{-\left(\frac{n\pi}{a}\right)^2 t} \cdot \sin \frac{n\pi}{a} \cdot x$$

apply the condition $w(0) = x$

$$x = \sum_{n=1}^{\infty} b_n \, e^{-\left(\frac{n\pi}{a}\right)^2 \cdot 0} \sin\left(\frac{n\pi}{a}\right) \cdot x$$

$$x = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{a} x$$

$$\Rightarrow b_n = \frac{2}{a} \int_0^a x \cdot \sin \frac{n\pi}{a} x \, dx$$

$$b_n = \frac{2}{a} \left[\left(\frac{-a}{n\pi} \cos \frac{n\pi}{a} x \right) x \Big|_0^a + \frac{a^2}{n^2\pi^2} \sin \frac{n\pi}{a} x \Big|_0^a \right]$$

$$\begin{array}{c|c} x & \sin \frac{n\pi}{a} x \\ 1 & -\frac{a}{n\pi} \cos \frac{n\pi}{a} x \\ 0 & -\frac{a^2}{n^2\pi^2} \sin \frac{n\pi}{a} x \end{array}$$

$$b_n = \frac{2}{a} \left[\left(\frac{-a^2}{n\pi} \cos n\pi - 0 \right) + \left(\frac{a^2}{n^2\pi^2} \sin n\pi - 0 \right) \right]$$

$$b_n = \frac{2}{a} \left[\frac{-a^2}{n\pi} (-1)^n + 0 \right]$$

$$\boxed{b_n = \frac{-2a}{n\pi} (-1)^n}$$

$$\therefore U(x,t) = \sum_{n=1}^{\infty} \frac{-2a(-1)^n}{n\pi} e^{-\left(\frac{n\pi}{a}\right)^2 t} \sin \frac{n\pi}{a} x$$

$$\boxed{U(x,t) = \frac{2a}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} e^{-\left(\frac{n\pi}{a}\right)^2 t} \sin \frac{n\pi}{a} x}$$

(Q V) $U_{xx} - 4U_{xy} + 4U_{yy} = \cos(2x+y) \rightarrow (1)$

(1) $\Phi = B^2 - 4AC = (-4)^2 - 4(1)(4) = 16 - 16 = 0$

It is parabolic

(2) $m = \frac{B}{2A} = \frac{-4}{2(1)} = -2$

$\frac{dy}{dx} = -2 \Rightarrow dy = -2\,dx$

$y = -2x + c_1 \Rightarrow \boxed{y + 2x = c_1}$

$\Rightarrow \eta = y + 2x$

choose $\xi = x$ such that $\begin{vmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{vmatrix} \neq 0$

$\begin{vmatrix} 1 & 0 \\ +2 & 1 \end{vmatrix} = 1 - 0 = 1 \neq 0$

$\xi = x$

$\eta = y + 2x$

$U_x = U_\xi \xi_x + U_\eta \eta_x = U_\xi + 2U_\eta$

$U_{xx} = U_{\xi\xi}\xi_x + U_{\xi\eta}\eta_x + 2U_{\eta\xi}\xi_x + 2U_{\eta\eta}\eta_x$

$U_{xx} = U_{\xi\xi} + 2U_{\xi\eta} + 2U_{\xi\eta} + 4U_{\eta\eta}$

$U_{xx} = U_{\xi\xi} + 4U_{\xi\eta} + 4U_{\eta\eta}$

$U_{xy} = \underset{0}{\cancel{U_{\xi\xi}\xi_y}} + U_{\xi\eta}\eta_y + \underset{0}{\cancel{2U_{\eta\xi}\xi_y}} + 2U_{\eta\eta}\eta_y$

$$U_{xy} = U_{\xi\eta} + 2U_{\eta\eta}$$

$$U_y = \underbrace{\cancel{U_\xi \xi_y}}_{0} + U_\eta \eta_y \Rightarrow U_y = U_\eta$$

$$U_{yy} = \underbrace{\cancel{U_{\eta\xi}\xi_y}}_{0} + U_{\eta\eta}\eta_y = U_{\eta\eta}$$

substitute in (1)

$$U_{\xi\xi} + 4U_{\xi\eta} + 4U_{\eta\eta} - 4(U_{\xi\eta} + 2U_{\eta\eta}) + 4U_{\eta\eta} = \cos(\eta)$$

$$U_{\xi\xi} + \cancel{4U_{\xi\eta}} + \cancel{4U_{\eta\eta}} - \cancel{4U_{\xi\eta}} - \cancel{8U_{\eta\eta}} + \cancel{4U_{\eta\eta}} = \cos(\eta)$$

$$U_{\xi\xi} = \cos(\eta)$$

(3) integrate w.r.t (ξ)

$$U_\xi = \cos(\eta)\,\xi + f(\eta)$$

integrate w.r.t ξ again

$$U(\xi,\eta) = \cos(\eta)\frac{\xi^2}{2} + f(\eta)\cdot\xi + g(\eta)$$

$$\boxed{U(x,y) = \cos(y+2x)\frac{y^2}{2} + f(2x+y)\,y + g(2x+y)}$$