425 _ Math :

Question I :

A By lagrange method.

$$\frac{dx}{1} = \frac{dy}{1} = \frac{dz}{1} = \frac{dt}{1} \quad \text{and} \quad du = 0$$

$\Rightarrow$ ①: $\boxed{u = c_1}$

② : $\frac{dx}{1} = \frac{dy}{1} \Rightarrow \boxed{x - y = c_2}$

③ $\frac{dy}{1} = \frac{dz}{1} \Rightarrow \boxed{y - z = c_3}$

④ $\frac{dz}{1} = \frac{dt}{1} \Rightarrow \boxed{z - t = c_4}$

The general solution is

$$F(u, x-y, y-z, z-t) = 0$$

or

$$\boxed{u = f(x-y, y-z, z-t)}$$

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(B)

$$P(x) = u^2 - 2uy - y^2 \quad , \quad Q(x) = xy + xu$$
$$R(x) = xy - xu$$

$$\Rightarrow \quad \frac{dx}{u^2-2uy-y^2} = \frac{dy}{xy+xu} = \frac{du}{xy-xu}$$

(1) since : $x(u^2-2uy-y^2) + y(xy+xu) + u(xy-xu) = 0$
then :

$$x\,dx + y\,dy + u\,du = 0$$
$$\Rightarrow \boxed{x^2 + y^2 + u^2 = C_1}$$

(2) $$\frac{dy}{\cancel{x}(y+u)} = \frac{du}{\cancel{x}(y-u)} \Rightarrow (y-u)\,dy = (y+u)\,du$$

$$\Rightarrow y\,dy - u\,dy = y\,du + u\,du$$

$$\Rightarrow y\,dy - u\,du = y\,du + u\,dy = d(uy)$$
$$\frac{y^2}{2} - \frac{u^2}{2} = uy + C$$

$$\Rightarrow \boxed{y^2 - u^2 - 2uy = C_2} \qquad C_2 = 2C$$

$$F(x^2+y^2+u^2 \,;\, y^2-u^2-2uy) = 0$$

or

$$x^2+y^2+u^2 = f(y^2-u^2-2uy)$$

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Question II

(A) ① $\Delta = B^2 - 4AC = 4 - 4(1)(1) = 0$

The equation (1) is parabolic.

(b) $\xi = x \qquad \eta = x - y$

$$\begin{tikzcd}[row sep=small] & & x \\ & \xi \arrow[ur] \arrow[r] & y \\ u \arrow[ur] \arrow[dr] & & x \\ & \eta \arrow[ur] \arrow[r] & y \end{tikzcd}$$

$$u_x = u_\xi \xi_x + u_\eta \eta_x$$

$$u_x = u_\xi + u_\eta$$

$$u_{xx} = u_{\xi\xi} \xi_x + u_{\xi\eta} \eta_x + u_{\eta\xi} \xi_x + u_{\eta\eta} \eta_x$$

$$\boxed{u_{xx} = u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}}$$

$$u_{xy} = u_{\xi\xi} \underbrace{\xi_y}_{0} + u_{\xi\eta} \eta_y + u_{\eta\xi} \underbrace{\xi_y}_{0} + u_{\eta\eta} \eta_y$$

$$\boxed{u_{xy} = -u_{\xi\eta} - u_{\eta\eta}}$$

$$u_y = u_\xi \xi_y + u_\eta \eta_y = -u_\eta$$

$$u_{yy} = -u_{\eta\xi} \underbrace{\xi_y}_{0} - u_{\eta\eta} \eta_y \Rightarrow u_{yy} = u_{\eta\eta}$$

substitute in ① $u_{\xi\xi} + \cancel{2u_{\xi\eta}} + \cancel{u_{\eta\eta}} - \cancel{2u_{\xi\eta}} - \cancel{2u_{\eta\eta}} + \cancel{u_{\eta\eta}} = 0$

$$\Rightarrow \boxed{u_{\xi\xi} = 0}$$

(c) $L = \frac{\partial^2}{\partial x^2} + 2\frac{\partial}{\partial x}\frac{\partial}{\partial y} + \frac{\partial^2}{\partial y^2}$

$= \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)$

$= \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)^2$

The general solution is given by.

$$\boxed{u(x,y) = x f(x-y) + g(x-y)}$$

or $U_{\xi\xi} = 0$

$U_\xi = f(\eta)$

$U = f(\eta)\xi + g(\eta)$

$U = f(x-y)x + g(x-y)$

=

(B) The harmonic function is satisfying the Lapalce eq.

$$U_{xx} + U_{yy} + U_{zz} = 0$$

Let $w = ax + by + cz + d$

$U_x = a f_w \Rightarrow U_{xx} = a^2 f_{ww}$

$U_y = b f_w \Rightarrow U_{yy} = b^2 f_{ww}$

$U_z = c f_w \Rightarrow U_{zz} = c^2 f_{ww}$

$\Rightarrow (a^2 + b^2 + c^2) f_{ww} = 0$

we have two cases.

(1) $(a^2+b^2+c^2) = 0$ or (2) $f_{ww} = 0$

$\Rightarrow a = b = c = 0$

$\Rightarrow u$ is constant function

So $a^2 + b^2 + c^2 \neq 0$

could be any constants.

$\Rightarrow f$ is linear function

$f(w) = \alpha w + \beta$

when α and β are constants.

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Question III

A) $u_x + x u_y = \dfrac{e^u}{1+x}$

$$\frac{dx}{1} = \frac{dy}{x} = \frac{du}{\frac{e^u}{1+x}}$$

(1) $\dfrac{dx}{1} = \dfrac{dy}{x} \Rightarrow x\,dx = dy \Rightarrow \boxed{y - \dfrac{x^2}{2} = c_1}$

(2) $\dfrac{dx}{1} = \dfrac{du}{\frac{e^u}{1+x}} \Rightarrow \dfrac{dx}{1+x} = e^{-u}du$

$\Rightarrow \ln(1+x) = -e^{-u} + c_2 \Rightarrow \boxed{\ln(1+x) + e^{-u} = c_2}$

$\Rightarrow F\left(y - \dfrac{x^2}{2}, \ln(1+x) + e^{-u}\right) = 0$

$$\ln(1+x) + e^{-u} = f\left(y - \frac{x^2}{2}\right)$$

when $x=0$ $y=t$ $u=t$

$\underbrace{\ln 1}_{0} + e^{-t} = f(t) \Rightarrow f(t) = e^{-t}$

$\therefore \boxed{\ln(1+x) + e^{-u} = e^{\frac{x^2}{2} - y}}$

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(B) $\dfrac{dx}{x^2} = \dfrac{dy}{y^2} = \dfrac{du}{-u^2}$

(1) $\dfrac{dx}{x^2} = \dfrac{dy}{y^2} \Rightarrow \dfrac{-1}{x} = \dfrac{-1}{y} + c_1 \Rightarrow \boxed{\dfrac{1}{y} - \dfrac{1}{x} = c_1}$

(2) $\dfrac{dy}{y^2} = \dfrac{du}{-u^2} \Rightarrow \dfrac{-1}{y} = +\dfrac{1}{u} + c \Rightarrow \boxed{\dfrac{1}{u} + \dfrac{1}{y} = c_2}$

$$F\left(\frac{1}{y} - \frac{1}{x}, \frac{1}{u} + \frac{1}{y}\right) = 0$$

$$\frac{1}{y} - \frac{1}{x} = f\left(\frac{1}{u} + \frac{1}{y}\right)$$

$$\frac{x-y}{xy} = f\left(\frac{1}{u} + \frac{1}{y}\right)$$

let $x = t$

$\Rightarrow yt = t + y$

$\Rightarrow yt - y = t$

$\Rightarrow y(t-1) = t$

$\Rightarrow \boxed{y = \dfrac{t}{t-1}}$

$$\Rightarrow \frac{t - \frac{t}{t-1}}{\frac{t^2}{t-1}} = f\left(1 + \frac{t-1}{t}\right)$$

$$\Rightarrow \frac{t^2 - t - t}{t^2} = f\left(\frac{2t-1}{t}\right)$$

$$\Rightarrow \frac{t-2}{t} = f\left(\frac{2t-1}{t}\right)$$

$w = \dfrac{2t-1}{t} \Rightarrow wt = 2t - 1 \Rightarrow wt - 2t = -1 \Rightarrow t(w-2) = -1$

$$\boxed{t = \frac{-1}{w-2}}$$

$$f(w) = \frac{\frac{-1}{w-2} - 2}{\frac{-1}{w-2}} = \frac{-1 - 2w + 4}{-1} = 2w - 3$$

$\therefore \dfrac{x-y}{xy} = 2\left(\dfrac{1}{u} + \dfrac{1}{y}\right) - 3 \Rightarrow \dfrac{x-y}{xy} = \dfrac{2(y+u)}{yu} - 3\,\dfrac{uy}{uy}$

$(x-y)yu = xy(2(y+u) - 3uy)$

$xyu - y^2u = 2xy^2 + 2xyu - 3uxy^2 \Rightarrow \boxed{u(3y^2x - y^2 - xy) = 2xy^2}$

(c) (1) $U_{xx} - 4U_{yy} = 0$

$$L = \left(\frac{\partial^2}{\partial x^2} - 4\frac{\partial^2}{\partial y^2}\right) = \left(\frac{\partial}{\partial x} - 2\frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial x} + 2\frac{\partial}{\partial y}\right)$$

$\Rightarrow a_1 = 1 ; b_1 = -2 , c_1 = 0$ & $a_2 = 1 , b_2 = 2 , c_2 = 0$

$\Rightarrow U(x,y) = f(-2x-y) + g(2x-y)$

$$\boxed{U_h(x,y) = f(2x+y) + g(2x-y)}$$

(2) Let $U_p = A\sin(x+y) + B\cos(x+y)$

$(U_p)_x = A\cos(x+y) - B\sin(x+y)$

$(U_p)_{xx} = -A\sin(x+y) - B\cos(x+y)$

$(U_p)_y = A\cos(x+y) - B\sin(x+y)$

$(U_p)_{yy} = -A\sin(x+y) - B\cos(x+y)$

$\Rightarrow -A\sin(x+y) - B\cos(x+y) + 4A\sin(x+y) + 4B\cos(x+y) = \sin(x+y)$

$\Rightarrow 3A\sin(x+y) + 3B\cos(x+y) = \sin(x+y)$

$\Rightarrow 3A = 1 \Rightarrow \boxed{A = \frac{1}{3}}$ & $3B = 0 \Rightarrow \boxed{B = 0}$

$U_p = \frac{1}{3}\sin(x+y)$

$\therefore U(x,y) = f(2x+y) + g(2x-y) + \frac{1}{3}\sin(x+y)$

$U_x(x,y) = 2f'(2x+y) + 2g'(2x-y) + \frac{1}{3}\cos(x+y)$

- $U(0,y) = 0 \Rightarrow \boxed{f(y) + g(-y) + \frac{1}{3}\sin y = 0} \rightarrow (I)$

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$\Rightarrow u_x(0,y)=0$

$\Rightarrow \boxed{2f'(y)+2g'(-y)+\frac{1}{3}\cos y=0} \rightarrow (2)$

$\Rightarrow 2f(y)-2g(-y)+\frac{1}{3}\sin y=C$

add with (1×2)

$2f(y)+2g(-y)+\frac{2}{3}\sin y=0$

$4f(y)+\sin y=C$

$f(y)=\frac{1}{4}(C-\sin y)$

sub in (1)

$\frac{1}{4}(C-\sin y)+g(-y)+\frac{1}{3}\sin y=0$

$g(-y)=-\frac{1}{3}\sin y-\frac{1}{4}(C-\sin y)$

$g(-y)=\left(-\frac{1}{3}+\frac{1}{4}\right)\sin y-\frac{1}{4}C$

$g(-y)=\frac{-1}{12}\sin y-\frac{1}{4}C$

$\Rightarrow g(y)=\frac{-1}{12}\sin(-y)-\frac{1}{4}C$

$g(y)=\frac{1}{12}\sin(y)-\frac{1}{4}C$

$\therefore U(x,y)=\frac{1}{4}\left(\cancel{C}-\sin(2x+y)\right)+\frac{1}{12}\sin(2x-y)-\cancel{\frac{1}{4}C}$

$+\frac{1}{3}\sin(x+y)$

$$\boxed{U(x,y)=\frac{-1}{4}\sin(2x+y)+\frac{1}{12}\sin(2x-y)+\frac{1}{3}\sin(x+y)}$$

Question IV :

$$U_t = 2U_{xx} \qquad 0<x<\pi \quad t>0$$

$$u(0,t) = 5 \quad , \quad u(\pi,t) = 10 \qquad t>0$$
$$u(x,0) = \sin 3x - \sin 5x \qquad 0<x<\pi$$

Here $T_0 = 5 \qquad T_1 = 10$

$$\Rightarrow \psi(x) = \frac{T_1 - T_0}{e} x + T_0$$

$$\Rightarrow \boxed{\psi(x) = \frac{5}{\pi} x + 5}$$

Now v satisfies the system

$$v_t = 2v_{xx} \qquad 0<x<\pi \ , \ t>0$$
$$v(0,t) = v(\pi,t) = 0 \qquad t>0$$
$$v(x,0) = \sin 3x - \sin 5x - \frac{5}{\pi}x - 5 \ , \quad 0<x<\pi$$

To find v

$$\Rightarrow v(x,t) = X(x)\,Y(t)$$

$$XY_t = 2X_{xx}Y$$

$\div 2XY$

$$\frac{Y_t}{2Y} = \frac{X_{xx}}{X} = -\lambda^2$$

① $Y_t + 2\lambda^2 Y = 0$

By integration factor $e^{\int 2\lambda^2 dt} = e^{2\lambda^2 t}$

$$Y_t \cdot e^{2\lambda^2 t} + 2\lambda^2 e^{2\lambda^2 t} Y = 0$$

$$\frac{d}{dt}\left(Y e^{2\lambda^2 t}\right) = 0$$

$$\Rightarrow Y e^{2\lambda^2 t} = C$$

$$\boxed{Y = Ce^{-2\lambda^2 t}}$$

② $X_{xx} + \lambda^2 X = 0$

$m^2 + \lambda^2 = 0 \Rightarrow m^2 = -\lambda^2 \Rightarrow m = \pm\sqrt{-\lambda^2}$

$m = \pm i\lambda$

$$\boxed{X(x) = a\cos\lambda x + b\sin\lambda x}$$

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Apply the conditions.

$V(0,t)=0 \Rightarrow X(0)=0 \Rightarrow \boxed{a=0}$

$V(\pi,t)=0 \Rightarrow X(\pi)=0 \Rightarrow b\sin\lambda\pi=0$

since $b\neq 0 \Rightarrow \sin\lambda\pi = 0$

$\Rightarrow \lambda\pi = n\pi \Rightarrow \lambda=\frac{n\pi}{\pi} \Rightarrow \boxed{\lambda_n=n}$

$n\in\mathbb{N}$

The general solution :

$$V(x,t)=\sum_{n=1}^{\infty} B_n e^{-2\lambda_n^2 t}\cdot\sin\lambda_n x$$

$$V(x,t)=\sum_{n=1}^{\infty} B_n e^{-2n^2 t}\cdot\sin nx$$

- $V(x,0)=\sin 3x-\sin 5x-\frac{5}{\pi}x-5$

$$\sin 3x-\sin 5x-\frac{5}{\pi}x-5=\sum_{n=1}^{\infty}B_n\sin nx$$

$$B_n=\frac{2}{\pi}\int_0^{\pi}\left(\sin 3x-\sin 5x-\frac{5}{\pi}x-5\right)\sin nx\,dx$$

$$B_n=\frac{2}{\pi}\left[\int_0^{\pi}\sin 3x\sin nx\,dx-\int_0^{\pi}\sin 5x\sin nx\,dx-\int_0^{\pi}\left(\frac{5}{\pi}x+5\right)\sin nx\,dx\right.$$

$$B_n=\frac{2}{\pi}\left[\int_0^{\pi}\frac{1}{2}\left(\cos(3-n)x-\cos(n+3)x\right)dx+\int_0^{\pi}\frac{1}{2}\left(\cos(5-n)x-\cos(n+5)x\right)dx\right.$$

$$-\int_0^{\pi}\left(\frac{5}{\pi}x+5\right)\sin nx\,dx$$

by Parts

$u=\frac{5}{\pi}x+5 \quad dv=\sin nx\,dx$

$du=\frac{5}{\pi}dx \quad V=\frac{-\cos nx}{n}$

$n\neq 3$

$n\neq 5$

$$B_n=\frac{2}{\pi}\left[\frac{1}{2}\frac{\sin(3-n)x}{3-n}\Big|_0^{\pi}-\frac{1}{2}\frac{\sin(n+3)x}{n+3}\Big|_0^{\pi}+\frac{1}{2}\frac{\sin(5-n)x}{5-n}\Big|_0^{\pi}+\right.$$

$$\left.\frac{1}{2}\frac{\sin(5+n)}{5+n}\Big|_0^{\pi}+\frac{\left(\frac{5}{\pi}x+5\right)\cos nx}{n}\Big|_0^{\pi}-\frac{5}{n\pi}\int_0^{\pi}\cos nx\,dx\right]$$

$$B_n = \frac{1}{\pi}(0-0) - \frac{1}{\pi}(0-0) + \frac{1}{\pi}(0-0) + \frac{1}{\pi}(0-0)$$

$$\left(\frac{2}{\pi}\left(\frac{10}{n}\left((\cos n\pi) - \frac{5}{n}\cos 0\right)\right) - \frac{25}{\pi / n\pi}\left(\underbrace{\frac{\sin nx}{n}}_{0''}\Big|_0^{\pi}\right)\right)$$

$$\Rightarrow B_n = \frac{2}{\pi}\left(\frac{10}{n}(-1)^n - \frac{5}{n}\right) = \left(\frac{5}{n}\left(2(-1)^n - 1\right)\right)\frac{2}{\pi}$$

$$B_1 = \cancel{-15}\ \frac{-30}{\pi}$$

$$B_2 = \frac{2}{\pi}\left(\frac{5}{2}\left(2(+1)-1\right)\right) = \frac{5}{\pi}$$

$$\underbrace{B_3 = 1}, \quad \underbrace{B_5 = 1}, \quad B_4 = \frac{2}{\pi}\left(\frac{5}{4}(2-1)\right) = \frac{5}{2\pi}$$

by comparing.

...

$$V(x,y) = \cancel{\sum_n} \frac{-30}{\pi} e^{-2t}\sin x + \frac{5}{\pi} e^{-8t}\sin 2x$$

$\therefore$ ~~u(x,y) =~~

$$+ e^{-18t}\sin 3x + \cdots$$

$$\boxed{u(x,y) = v(x,y) + \frac{5}{\pi}x + 5}$$

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(B) Assume that the equation have two solutions u_1 & u_2 and let $v = u_1 - u_2$

$$\Rightarrow \quad \frac{\partial v}{\partial t} = \beta \frac{\partial^2 v}{\partial x^2}$$

$$v(0,t) = v(L,t) = 0$$

$$v(x,0) = F(x) - F(x) = 0$$

Define the potential function:

$$E(t) = \frac{1}{2\beta} \int_0^L v^2(x,t)\,dx$$

$$\Rightarrow E'(t) = \frac{1}{2\beta} \int_0^L 2v(x,t) \cdot v_t(x,t)\,dx$$

$$= \frac{1}{\cancel{2\beta}} \int_0^L \cancel{2\beta}\, v \; v_{xx}\,dx$$

$$E'(t) = \int_0^L v\,v_{xx}\,dx \quad \text{by parts}$$

$$E'(t) = v_x v \Big|_0^L - \int_0^L v_x^2\,dx$$

$$= v_x(L,t)\underbrace{v(L,t)}_{0} - v_x(0,t)\underbrace{v(0,t)}_{0} - \int_0^L v_x^2\,dx$$

$$E'(t) = -\int_0^L v_x^2\,dx \le 0$$

but $E(0) = \int_0^L \frac{1}{2\beta} \underbrace{v^2(x,0)}_{0}\,dx = 0$

and we have $E(t) \geq 0$ & $E(0) = 0$ & $(E'(t) \le 0$

$$\Rightarrow E(t) \le 0$$

$$\Rightarrow E(t) = 0 \quad \forall t > 0 \quad \Rightarrow v(x,t) \equiv 0$$

$$\Rightarrow u_1 = u_2.$$

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