لا يكتب في
هذا الهامش

Q I (A)

subsidiary equations.

$$\frac{dx}{P} = \frac{dy}{Q}, \quad \frac{dy}{Q} = \frac{dz}{R}$$

The solution of these two equation are

$$v(x,y,z) = c_1 \quad \& \quad w(x,y,z) = c_2$$

$$\Rightarrow v_x dx + v_y dy + v_z dz = 0 \quad \& \quad w_x dx + w_y dy + w_z dz = 0$$

$\div dx$

$$v_x + v_y \frac{dy}{dx} + v_z \frac{dz}{dx} = 0 \quad \& \quad w_x + w_y \frac{dy}{dx} + w_z \frac{dz}{dx} = 0$$

$$v_x + v_y \frac{Q}{P} + v_z \frac{R}{P} = 0 \quad \& \quad w_x + w_y \frac{Q}{P} + w_z \frac{R}{P} = 0$$

$$P v_x + Q v_y + R v_z = 0 \rightarrow (1) \quad \& \quad P w_x + Q w_y + R w_z = 0 \rightarrow (2)$$

multiply equations (1), (2) by w_z & v_z respectively

$$P v_x w_z + Q v_y w_z + R v_z w_z = 0$$

$$P w_x v_z + Q w_y v_z + R w_z v_z = 0$$

substracted these two equations

$$P(w_x v_z - v_x w_z) = Q(v_y w_z - w_y v_z)$$

$$\Rightarrow \frac{P}{(v_y w_z - w_y v_z)} = \frac{Q}{w_x v_z - v_x w_z} \Rightarrow \frac{P}{\begin{vmatrix} v_y & v_z \\ w_y & w_z \end{vmatrix}} = \frac{Q}{\begin{vmatrix} v_z & v_x \\ w_z & w_x \end{vmatrix}} \rightarrow (A)$$

Similarly $$\frac{Q}{\begin{vmatrix} v_z & v_x \\ w_z & w_x \end{vmatrix}} = \frac{R}{\begin{vmatrix} v_x & v_y \\ w_x & w_y \end{vmatrix}} \rightarrow (B)$$

from (A) & (B) $$\frac{P}{\begin{vmatrix} v_y & v_z \\ w_y & w_z \end{vmatrix}} = \frac{Q}{\begin{vmatrix} v_z & v_x \\ w_z & w_x \end{vmatrix}} = \frac{R}{\begin{vmatrix} v_x & v_y \\ w_x & w_y \end{vmatrix}}$$

$$\Rightarrow \frac{\partial(v,w)}{\partial(z,x)} = \frac{Q}{P}\frac{\partial(v,w)}{\partial(y,z)}, \quad \frac{\partial(v,w)}{\partial(x,y)} = \frac{R}{P}\frac{\partial(v,w)}{\partial(y,z)}$$

لا يكتب في
هذا الهامش

Q I (B) $u(x+u)u_x - y(y+u)u_y = 0$

$$\frac{dx}{u(x+u)} = \frac{dy}{-y(y+u)}, \quad du = 0 \Rightarrow \boxed{u = c_1}$$

$$\Rightarrow \frac{dx}{c_1(x+c_1)} = \frac{dy}{-y(y+c_1)}, \quad \frac{-1}{y(y+c_1)} = \frac{\frac{1}{c_1}}{y} - \frac{\frac{1}{c_1}}{y+c_1}$$

$$\frac{dx}{c_1(x+c_1)} = \left[\frac{\frac{1}{c_1}}{y} - \frac{\frac{1}{c_1}}{y+c_1}\right] dy$$

$$\frac{1}{c_1}\ln|x+c_1| = \frac{1}{c_1}\ln|y| - \frac{1}{c_1}\ln|y+c_1| + \ln C_2$$

$$\ln(x+c_1) - \ln(y) + \ln(y+c_1) = \ln C_2$$

$$\ln\left(\frac{(x+c_1)(y+c_1)}{y}\right) = \ln C_2$$

$$\frac{(x+c_1)(y+c_1)}{y} = C_2$$

$$\Rightarrow \frac{(x+u)(y+u)}{y} = C_2$$

$$\therefore v = u \qquad w = \frac{(x+u)(y+u)}{y}$$

$$F(v, w) = 0$$

$$\Rightarrow f(u) = \left\{\frac{(x+u)(y+u)}{y}\right\} \qquad \begin{matrix} x=1 \\ y=t \\ u=\sqrt{t} \end{matrix}$$

$$f(\sqrt{t}) = \frac{(1+\sqrt{t})(t+\sqrt{t})}{t}$$

$$f(w) = \frac{(1+w)(w^2+w)}{w^2} \qquad \sqrt{t} = w \Rightarrow w^2 = t$$

$$f(w) = \frac{(1+w)^2}{w}$$

$$\therefore \frac{(1+u)^2}{u} = \frac{(x+u)(y+u)}{y} \Rightarrow \boxed{y(1+u)^2 = (x+u)(y+u)\,u}$$

لا يكتب في
هذا الهامش

Q I (C) 1. $(u^2-2uy-y^2)u_x+(xy+xu)u_y = xy-xu$

$$\frac{dx}{u^2-2uy-y^2} = \frac{dy}{xy+xu} = \frac{du}{xy-xu}$$

① $\dfrac{dy}{\cancel{x}(u+y)} = \dfrac{du}{\cancel{x}(y-u)} \Rightarrow (y-u)\,dy = (u+y)\,du$

$\Rightarrow y\,dy = u\,dy + y\,du + u\,du$

$\frac{y^2}{2} = d(uy) + \frac{u^2}{2} + C \Rightarrow y^2 = 2uy + u^2 + 2C$

$\Rightarrow u^2 - y^2 + 2uy = C_1 \quad (C_1 = -2C)$

② since $x(u^2-2uy-y^2) + y(xy+xu) + u(xy-xu)$
$= \cancel{u^2x} - \cancel{2xuy} - \cancel{xy^2} + \cancel{xy^2} + \cancel{xuy} + \cancel{xuy} - \cancel{xu^2} = 0$

$\Rightarrow x\,dx + y\,dy + u\,du = 0$

$\Rightarrow x^2 + y^2 + u^2 = C_2$

$$F(u^2-y^2+2uy,\ x^2+y^2+u^2) = 0$$

$$x^2+y^2+u^2 = f(u^2-y^2+2uy)$$

2. $(x-y)u_{xy} - u_x + u_y = 0 \to$ ① $\quad (V = (x-y)u)$

~~$u_x = \frac{1}{x-y}$~~ $\quad V_x = (x-y)u_x + u$ [redacted]

$V_y = \cancel{u_x} - u + (x-y)u_y$

$V_{xy} = -u_x + (x-y)u_{xy} + u_y$

$(x-y)u_{xy} = V_{xy} + u_x - u_y$

sub. in ①

$V_{xy} + \cancel{u_x} - \cancel{u_y} - \cancel{u_x} + \cancel{u_y} = 0 \Rightarrow V_{xy} = 0$

$V_x = f(x) \Rightarrow V = F(x) + g(y) \Rightarrow u = \dfrac{F(x)+g(y)}{x-y}$

$F'(x) = f(x)$

لا يكتب في هذا الهامش

Q II (A) $4U_{xx} + 12U_{xy} + 9U_{yy} = 0 \rightarrow (*)$

$\phi = B^2 - 4AC = (12)^2 - 4(4)(9) = 144 - 144 = 0$

$\therefore$ the equation is parabolic.

(B) $\dfrac{dy}{dx} = \dfrac{B}{2A} = \dfrac{12}{2(4)} = \dfrac{3}{2}$

$2dy = 3dx \Rightarrow 2y - 3x = C_1$

choose $\eta = 2y - 3x$ & $\xi = x$ such that

$$\begin{vmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ -3 & 2 \end{vmatrix} = 2 \neq 0$$

$\eta_x = -3 \ , \ \eta_y = 2 \ , \ \eta_{xx} = \eta_{yy} = \eta_{xy} = 0$

$\xi_x = 1 \ , \ \xi_y = \xi_{xx} = \xi_{yy} = \xi_{xy} = 0$

$U_x = U_\xi \xi_x + U_\eta \eta_x \Rightarrow \boxed{U_x = U_\xi - 3U_\eta}$

$U_y = U_\xi \xi_y + U_\eta \eta_y \Rightarrow \boxed{U_y = 2U_\eta}$

$U_{xx} = U_{\xi\xi}\xi_x + U_{\xi\eta}\eta_x - 3(U_{\xi\eta}\xi_x + U_{\eta\eta}\eta_x)$

$U_{xx} = U_{\xi\xi} - 3U_{\xi\eta} - 3U_{\xi\eta} + 9U_{\eta\eta} \Rightarrow \boxed{U_{xx} = U_{\xi\xi} - 6U_{\xi\eta} + 9U_{\eta\eta}}$

$U_{yy} = 2(U_{\xi\eta}\xi_y + U_{\eta\eta}\eta_y) = 4U_{\eta\eta} \Rightarrow \boxed{U_{yy} = 4U_{\eta\eta}}$

$U_{xy} = 2(U_{\xi\eta}\xi_x + U_{\eta\eta}\eta_x) \Rightarrow \boxed{U_{xy} = 2U_{\xi\eta} - 6U_{\eta\eta}}$

substitute in (*)

(C) $4U_{\xi\xi} - \cancel{24U_{\xi\eta}} + \cancel{36U_{\eta\eta}} + \cancel{24U_{\xi\eta}} - \cancel{72U_{\eta\eta}} + \cancel{36U_{\eta\eta}} = 0$

$\Rightarrow U_{\xi\xi} = 0 \Rightarrow U_\xi = f(\eta) \Rightarrow U(\xi,\eta) = \xi f(\eta) + g(\eta)$

$\therefore \boxed{U(x,y) = x f(2y-3x) + g(2y-3x)}$

لا يكتب في
هذا الهامش

Q II (D)

$$a(x^2+y^2)+bz^2=1$$

$$a(2x)+b\,2zz_x=0$$
$$a(2y)+b\,2zz_y=0$$

$$\Rightarrow \quad 2a+2b(z_x^2+zz_{xx})=0$$
$$2a+2b(z_y^2+zz_{yy})=0$$

substract:

$$2b(z_x^2+zz_{xx}-z_y^2-zz_{yy})=0$$

$$b\neq 0 \Rightarrow z_x^2-z_y^2+z(z_{xx}-z_{yy})=0$$

لا يكتب في
هذا الهامش

Q III (A) $u_t = k u_{xx}$ $0<x<1$, $t>0$

let $U(x,t) = V(x)\,W(t)$

$\Rightarrow W_t \cdot V = k(W V_{xx})$

$\Rightarrow \dfrac{W_t}{kW} = \dfrac{V_{xx}}{V} = -\lambda^2$ ($-\lambda^2$ is a separation constant)

(1) $\Rightarrow W_t + kW\lambda^2 = 0$

integration factor : $e^{\int k\lambda^2 dt} = e^{k\lambda^2 t}$

$\Rightarrow W_t e^{k\lambda^2 t} + kW\lambda^2 e^{k\lambda^2 t} = 0$

$\dfrac{d}{dt}\left(W e^{k\lambda^2 t}\right) = 0$

$\Rightarrow W e^{k\lambda^2 t} = C_1$

$\Rightarrow \boxed{W(t) = C_1 e^{-k\lambda^2 t}}$

(2) $V_{xx} + \lambda^2 V = 0$

$m^2 + \lambda^2 = 0$

$m^2 = -\lambda^2 \Rightarrow m = \pm i\lambda$

$V(x) = a\cos\lambda x + b\sin\lambda x$ $\lambda \neq 0$

$V(x) = a + bx$ $\lambda = 0$

$\therefore U(x,t) = V(x) \cdot W(t)$

(B) $u(0,t) = u(1,t) = 0$

$u(x,0) = f(x)$

if $\lambda = 0 \Rightarrow \left.\begin{array}{l} V(0) = a = 0 \\ V(1) = b = 0 \end{array}\right\} \Rightarrow V = 0 \Rightarrow U = 0$ trivial solution

So $\lambda \neq 0$

$V(0) = 0 \Rightarrow \boxed{a = 0}$

$V(1)=0 \Rightarrow b\sin\lambda = 0$

$\Rightarrow \sin\lambda = 0 \Rightarrow \lambda = n\pi \quad n \in \mathbb{N}$

$$\boxed{\lambda_n = n\pi}$$

$\therefore U_n(x,t) = A_n e^{-\lambda^2 k t} \cdot \sin n\pi x$

The general solution of heat equations:

$$U(x,t) = \sum_{n=1}^{\infty} A_n e^{-k\lambda^2 t} \cdot \sin n\pi x \longrightarrow (**)$$

apply the condition $u(x,0) = f(x)$

$$f(x) = \sum_{n=1}^{\infty} A_n \sin n\pi x$$

$$\Rightarrow A_n = \frac{1}{1}\int_0^1 f(x) \sin n\pi x \, dx$$

$$A_n = \int_0^1 f(x) \cdot \sin n\pi x \, dx$$

$$\therefore u(x,t) = \sum_{n=1}^{\infty} \left(\int_0^1 f(x) \sin n\pi x \, dx \right) e^{-k\lambda_n^2 t} \sin n\pi x$$

لا يكتب في
هذا الهامش

QIV

(A) $u_{tt} = 4u_{xx}$ $\quad 0<x<\pi \quad t>0$

$u(x,0)=0$, $u_t(x,0)=x$

$u(0,t)=u(\pi,t)=0 \quad t>0$

$\Rightarrow$ let $u(x,t)=V(x)\,W(t)$

$\Rightarrow \cancel{V}W_{tt}=4V_{xx}W$

$$\frac{W_{tt}}{W}=\frac{4V_{xx}}{V}=-\lambda^2$$

$W_{tt}+4\lambda^2 W=0$	$\cancel{4}V_{xx}+\lambda^2 V=0$
$m^2+4\lambda^2=0$	$m^2+\lambda^2=0$
$m^2=-4\lambda^2$	$m=\pm i\lambda$
$m=\pm 2i\lambda$	
$\lambda\neq 0 \Rightarrow W(t)=a\cos 2\lambda t+b\sin 2\lambda t$	$V(x)=c\cos\lambda x+d\sin\lambda x \quad \lambda\neq 0$
$\lambda=0 \quad W(t)=a+bt$	$V(x)=c+dx \quad \lambda=0$

if $\lambda=0$ ~~with~~ $u(0,t)=0$ & $u(x,0)=0$

$\Rightarrow \boxed{c=0} \Rightarrow \boxed{a=0}$

$W_t=$ ~~[illegible]~~ b

$W_t(0)=0 \Rightarrow$ ~~[illegible]~~ $\Rightarrow b=0 \Rightarrow u=0$ (trivial)

So $\lambda\neq 0$

$u(0,t)=0$ & $u(x,0)=0$

$\Rightarrow V(0)=0$ & $W(0)=0$

$\Rightarrow \boxed{c=0}$ & $\boxed{a=0}$

$u(\pi,t)=0 \Rightarrow V(\pi)=0 \Rightarrow 0=d\sin\lambda\pi$

$\Rightarrow \sin\pi\lambda=0 \Rightarrow \lambda\pi=n\pi \quad n\in\mathbb{N}$

$\Rightarrow \boxed{\lambda_n=n}$

لا يكتب في هذا الهامش

The general solution:

$$U(x,t) = \sum_{n=1}^{\infty} A_n \sin 2nt \cdot \sin nx$$

$$U_t(x,t) = \sum_{n=1}^{\infty} A_n \cdot 2n \cos 2nt \cdot \sin nx$$

$$U_t(x,0) = x$$

$$x = \sum_{n=1}^{\infty} 2nA_n \cos(2n \cdot 0) \cdot \sin nx$$

$$x = \sum_{n=1}^{\infty} 2nA_n \sin nx$$

$$\Rightarrow 2nA_n = \frac{1}{\pi}\int_0^{\pi} x \cdot \sin nx \, dx$$

$$\int_0^{\pi} x \cdot \sin nx \, dx$$

$$\Rightarrow 2nA_n = \frac{1}{\pi}\left[\frac{-x\cos nx}{n}\Big|_0^{\pi} + \int_0^{\pi} \frac{\cos nx}{n} dx\right]$$

$u = x \qquad dv = \sin nx \, dx$

$du = dx \qquad v = \frac{-\cos nx}{n}$

$$2nA_n = \frac{1}{\pi}\left[\left(-\frac{\pi(-1)^n}{n} - 0\right) + \frac{\sin nx}{n^2}\Big|_0^{\pi}\right]$$

$$2nA_n = \frac{(-1)^{n+1}}{n} + \frac{1}{\pi n^2}\left[0\right]$$

$$2nA_n = \frac{(-1)^{n+1}}{n}$$

$$\Rightarrow \boxed{A_n = \frac{(-1)^{n+1}}{2n^2}}$$

$$\boxed{\therefore U(x,t) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n^2} \sin 2nt \cdot \sin nx}$$

لا يكتب في
هذا الهامش

Q IV

B

Assume that there are two solutions u_1 & u_2
and assume $V = u_1 - u_2$

$\Rightarrow V_{tt} = 4V_{xx}$

$V(x,0) = 0$, $V_t(x,0) = 0$
$V(0,t) = V(\pi,t) = 0$

let defined
$$E(t) = \frac{1}{2}\int_0^{\pi} \left((V_t)^2 + 4V_x^2 \right) dx \qquad t>0$$

this is called energy of function.

We note that $E(t) \geq 0 \rightarrow$ (1)

$$\Rightarrow E'(t) = \frac{1}{2}\int_0^{\pi} \Big(2V_t \underbrace{V_{tt}}_{\substack{\parallel \\ 4V_{xx}}} + 2\cdot 4\, V_x V_{xt}\Big)\, dx$$

$$E'(t) = \int_0^{\pi} (4V_t V_{xx} + 4V_x V_{xt})\, dx$$
$$= 4\int_0^{\pi} (V_t V_{xx} + V_x V_{xt})\, dx$$
$$= 4\int_0^{\pi} \frac{\partial}{\partial x}(V_t V_x)\, dx$$
$$= 4\, V_t V_x \Big|_0^{\pi}$$
$$= 4\big(V_t(\pi,t)\, V_x(\pi,t) - V_t(0,t)\, V_x(0,t)\big)$$
$$= 4(0) = 0$$
$\Rightarrow E(t)$ is a constant. $\rightarrow$ (2)

but $E(0) = 0 \rightarrow$ (3

لا يكتب في
هذا الهامش

So $E(t)=0 \quad \forall t$

$\Rightarrow V_x=0 \quad V_t=0$

$\Rightarrow V(x,t) = \text{constant}$

since $V(x_0,t)=0 \Rightarrow V=0$

$\Rightarrow u_1=u_2$