Introduction to Partial Differential Equations

General definition of partial differential equation, Classification, Sources, Solution.

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Outline

1. Background
2. Introduction
3. Classification
4. Sources of the PDES

- **Ordinary Differential Equations (ODEs)**
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- **Solution**: a function that satisfies the equation identically in some region.

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- L is linear if:

$$L(\alpha u + \beta v) = \alpha L(u) + \beta L(v)$$

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- If L is a differential operator and g is a function of the independent variables, then the differential equation $L(u) = g$ is homogenous if $g = 0$ and nonhomogenous otherwise.

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- Cauchy-Eular ODEs.

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  - $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$, (Laplace's equation).
  - $\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = 0$, (The wave equation).
  - $\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial y} = 0$, (The heat equation).
  - $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = g(x, y)$, (Poisson's equation)

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- We use subscript to mean differentiation with respect to the variables given, e.g. $u_t = \frac{\partial u}{\partial t}$.

Example (1)

- $xz_x - yz_y = \sin xy$
- $\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$
- $uu_y u_{xxx} + {u_{yy}}^2 = \sin u$
- $\left(\frac{\partial^2 u}{\partial x^2}\right)^2 + \frac{\partial^2 u}{\partial y^2} = \frac{\partial u}{\partial z} + z^3$
- $\begin{cases} w\frac{\partial v}{\partial x} + v\frac{\partial w}{\partial y} = x + y \\ v\frac{\partial v}{\partial x} + w\frac{\partial w}{\partial y} = x - y \end{cases}$
- $x\frac{\partial^2 u}{\partial t^2} + t\frac{\partial^2 u}{\partial y^2} + u^3\left(\frac{\partial u}{\partial x}\right)^2 = t + 1$

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- A PDE is said to be **linear** if it is of the first degree in the dependent variable (dependent variables when there is more than one dependent variable present) and the partial derivatives which occur in the equation, with coefficients depending only on the independent variables.
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- In Example (1) the first two equations are linear, while the remaining equations are nonlinear.

- The general form of a linear first-order equation in two independent variables is

$$P(x, y)z_x + Q(x, y)z_y + R(x, y)z = S(x, y)$$

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- The general form of a linear second-order equation in two independent variables is

$$\begin{aligned} A(x,y)z_{xx} + 2B(x,y)z_{xy} + C(x,y)z_{yy} + D(x,y)z_x \\ + E(x,y)z_y + F(x,y)z = G(x,y) \end{aligned}$$

- A PDE is called **quasilinear** if it is linear in the highest-order derivatives with coefficients depending on the independent variables, the unknown function and its derivatives of order lower than the order of the equation.
- The general form of a quasilinear first-order equation in two independent variables is

$$P(x, y, z)z_x + Q(x, y, z)z_y = R(x, y, z).$$

- A linear equation is certainly quasilinear, but in general the converse is not true.

Sources of the PDES

- The first source of DEs is the laws of physics that govern natural phenomena such as mechanical and electromagnetic motion and heat transfer.
- The second source is the result of eliminating arbitrary functions in the equations of surface.

Example

- $u(x, y) = f(2x + 3y)$
- $u(x, y, z) = x^n f(\frac{x}{y}, \frac{z}{y}), x \neq 0, y \neq 0$
- $u = f(2x + y) + g(x + 3y)$
- $u = f(x) + g(y)$