$\boxed{Q\,4.43}$ Let $\alpha(s)$ be a unit speed curve and β its involute

$\therefore \alpha$ is evolute of β. Let β be a plane curve.

claim: (α is a helix) ???

by Q(4.39) $\beta(s) = \alpha(s) + (c_1 - s)\, T_\alpha(s)$

Case 1 if $k_\alpha = 0$ then T_α = constant vector say it (u), So $\left[\begin{array}{l}\text{choose } u = T_\alpha \\ \text{axis of helix}\end{array}\right]$

$\langle T_\alpha, u\rangle = \langle T_\alpha, T_\alpha\rangle = 1 \Rightarrow \alpha$ is a helix

Case 2 if $k_\alpha > 0$, (β is a non-unit speed curve)

$$\frac{d\beta}{ds} = \overset{T_\alpha}{\alpha'(s)} + (c_1 - s)\, T_\alpha'(s) - T_\alpha(s) = (c_1 - s)\, k_\alpha N_\alpha .$$

$$\Rightarrow \left\| \frac{d\beta}{ds} \right\| = |c_1 - s|\, k_\alpha$$

Let $\bar{s}$ be arc length of $\beta \Rightarrow \bar{s} = \int \left\|\frac{d\beta}{ds}\right\| ds \Rightarrow \frac{d\bar{s}}{ds} = \left\|\frac{d\beta}{ds}\right\|$

$$\Rightarrow \boxed{\frac{d\bar{s}}{ds} = |c_1 - s|\, k_\alpha}$$

Now, $\dfrac{d\beta}{d\bar{s}} = \dfrac{d\beta}{ds} \cdot \dfrac{ds}{d\bar{s}} = (c_1 - s)\cancel{k_\alpha} N_\alpha \cdot \dfrac{1}{|c_1 - s|\cancel{k_\alpha}} = \pm N_\alpha$

and $T_\beta = \dfrac{d\beta/d\bar{s}}{\|d\beta/d\bar{s}\|} = \pm N_\alpha$ (لما الاشتقاق T_β بالنسبة لـ s لازم نعرف بالنسبة لـ $\bar{s}$)

$(N_\alpha' = -k_\alpha T_\alpha + \tau_\alpha B_\alpha)$

$$\frac{dT_\beta}{d\bar{s}} = \frac{dT_\beta}{ds} \cdot \frac{ds}{d\bar{s}} = \mp \left(\frac{dN_\alpha}{ds}\right)\frac{ds}{d\bar{s}} = \mp\left(-k_\alpha T_\alpha + \tau_\alpha B_\alpha\right)\frac{1}{|c_1 - s|\, k_\alpha}$$

$$\left\|\frac{dT_\beta}{d\bar{s}}\right\| = \frac{1}{|c_1 - s|\, k_\alpha}\sqrt{k_\alpha^2 + \tau_\alpha^2} \Rightarrow N_\beta = \frac{dT_\beta/d\bar{s}}{\|dT_\beta/d\bar{s}\|} = \frac{|c_1 - s|\, k_\alpha}{\sqrt{k_\alpha^2 + \tau_\alpha^2}} \pm \left[\frac{(-k_\alpha T_\alpha + \tau_\alpha B_\alpha)}{|c_1 - s|\, k_\alpha}\right]$$

$$\Rightarrow N_\beta = \mp \frac{(-k_\alpha T_\alpha + \tau_\alpha B_\alpha)}{\sqrt{k_\alpha^2 + \tau_\alpha^2}}$$

$$\text{Now, } B_\beta = T_\beta \times N_\beta = \mp \frac{1}{\sqrt{k_\alpha^2 + \tau_\alpha^2}}\left[N_\alpha \times (-k_\alpha T_\alpha + \tau_\alpha B_\alpha)\right] = \mp \frac{1}{\sqrt{k_\alpha^2 + \tau_\alpha^2}}\left[k_\alpha B_\alpha + \tau_\alpha (N_\alpha \times B_\alpha)\right]$$

$(B_\alpha = N_\alpha \times (-T_\alpha))$

$$= \pm \frac{k_\alpha}{k_\alpha\sqrt{1 + \left(\frac{\tau_\alpha}{k_\alpha}\right)^2}}\left[B_\alpha + \frac{\tau_\alpha}{k_\alpha} T_\alpha\right] \Rightarrow B_\beta = \mp \frac{1}{\sqrt{1 + \left(\frac{\tau_\alpha}{k_\alpha}\right)^2}}\left[B_\alpha + \frac{\tau_\alpha}{k_\alpha} T_\alpha\right]$$

(اسحب عامل مشترك k_α)

Now, find $B_\beta' = \dfrac{dB_\beta}{d\bar{s}} = \dfrac{dB_\beta}{ds} \cdot \dfrac{ds}{d\bar{s}}$ (as β is a plane curve $B_\beta' = 0$).

$$= \pm\left[\frac{1}{\sqrt{1 + \left(\frac{\tau_\alpha}{k_\alpha}\right)^2}}\left(B_\alpha' + \left(\frac{\tau_\alpha}{k_\alpha}\right)' T_\alpha + \left(\frac{\tau_\alpha}{k_\alpha}\right) T_\alpha'\right) + \left(\frac{-\left(\frac{\tau_\alpha}{k_\alpha}\right)\left(\frac{\tau_\alpha}{k_\alpha}\right)'}{\left[1 + \left(\frac{\tau_\alpha}{k_\alpha}\right)^2\right]^{3/2}}\left(B_\alpha + \frac{\tau_\alpha}{k_\alpha} T_\alpha\right)\right)\right]\frac{1}{|c_1 - s|\, k_\alpha} = 0$$

$$\text{Thus, } \frac{1}{\sqrt{1 + \left(\frac{\tau_\alpha}{k_\alpha}\right)^2}}\left[\left(-\tau_\alpha N_\alpha + \left(\frac{\tau_\alpha}{k_\alpha}\right)' T_\alpha + \left(\frac{\tau_\alpha}{k_\alpha}\right) k_\alpha N_\alpha\right) - \frac{\left(\frac{\tau_\alpha}{k_\alpha}\right)\left(\frac{\tau_\alpha}{k_\alpha}\right)'}{\left(1 + \left(\frac{\tau_\alpha}{k_\alpha}\right)^2\right)^{3/2}}\left(B_\alpha + \frac{\tau_\alpha}{k_\alpha} T_\alpha\right)\right] = 0$$

$$\frac{\left(\frac{\tau_\alpha}{k_\alpha}\right)'}{\sqrt{1 + \left(\frac{\tau_\alpha}{k_\alpha}\right)^2}} T_\alpha - \frac{\left(\frac{\tau_\alpha}{k_\alpha}\right)\left(\frac{\tau_\alpha}{k_\alpha}\right)'}{\left(1 + \left(\frac{\tau_\alpha}{k_\alpha}\right)^2\right)^{3/2}} B_\alpha - \frac{\left(\frac{\tau_\alpha}{k_\alpha}\right)^2\left(\frac{\tau_\alpha}{k_\alpha}\right)'}{\left(1 + \left(\frac{\tau_\alpha}{k_\alpha}\right)^2\right)^{3/2}} T_\alpha = 0.$$

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$$\Rightarrow \frac{\left(\frac{\tau_\alpha}{k_\alpha}\right)'}{\sqrt{1+\left(\frac{\tau_\alpha}{k_\alpha}\right)^2}}\left[T_\alpha - \frac{\frac{\tau_\alpha}{k_\alpha}}{1+\left(\frac{\tau_\alpha}{k_\alpha}\right)^2} B_\alpha - \frac{\frac{\tau_\alpha}{k_\alpha}}{1+\left(\frac{\tau_\alpha}{k_\alpha}\right)^2} T_\alpha\right] = 0$$

$$\Rightarrow \frac{\left(\frac{\tau_\alpha}{k_\alpha}\right)'}{\sqrt{1+\left(\frac{\tau_\alpha}{k_\alpha}\right)^2}}\left[\left(1 - \frac{\frac{\tau_\alpha}{k_\alpha}}{1+\left(\frac{\tau_\alpha}{k_\alpha}\right)^2}\right) T_\alpha - \frac{\frac{\tau_\alpha}{k_\alpha}}{1+\left(\frac{\tau_\alpha}{k_\alpha}\right)^2} B_\alpha\right] = 0$$

~~As~~

As $\{T_\alpha, B_\alpha\}$ are linearly Indpendent

$$\Rightarrow \frac{\left(\tau_\alpha/k_\alpha\right)'}{\sqrt{1+\left(\frac{\tau_\alpha}{k_\alpha}\right)^2}}\left(1 - \frac{\tau_\alpha/k_\alpha}{1+\left(\frac{\tau_\alpha}{k_\alpha}\right)^2}\right) = 0 \text{ and } \frac{\left(\tau_\alpha/k_\alpha\right)'\left(\tau_\alpha/k_\alpha\right)}{\left(1+\left(\frac{\tau_\alpha}{k_\alpha}\right)^2\right)^{3/2}} = 0 \to (*)$$

by $(*)$ $\quad 2\left(\frac{\tau_\alpha}{k_\alpha}\right)'\left(\frac{\tau_\alpha}{k_\alpha}\right) = 0 \Rightarrow 2\int\left(\frac{\tau_\alpha}{k_\alpha}\right)'\left(\frac{\tau_\alpha}{k_\alpha}\right) = 0$

$$\Rightarrow \left(\frac{\tau_\alpha}{k_\alpha}\right)^2 = c$$

$$\Rightarrow \frac{\tau_\alpha}{k_\alpha} = c$$

$\Rightarrow \tau_\alpha = c\, k_\alpha$

$\therefore$ α is a helix.