King Saud University

College of Science

Department of Mathematics

151 Math Exercises

(4.3)

Partial Orderings

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2018

Partial Orderings

DEFINITION 1 A relation R on a non empty set S is called a *partial ordering* or *partial order* if it is reflexive, antisymmetric, and transitive. A set S together with a partial ordering R is called a *partially ordered set*, or *poset*, and is denoted by (S,R). Members of S are called *elements* of the poset.

DEFINITION 2 Let a and b *be in the set S. Assume* (S,R) is a poset. We say that a and b are *comparable* if either $a\ R\ b$ or $b\ R\ a$. When a and b are elements of S such that neither $a\ R\ b$ nor $b\ R\ a$, a and b are called *incomparable*.

DEFINITION 3 If (S,R) is a poset and every two elements of S are comparable, S is called a ***totally ordered*** or ***linearly ordered set***, and R is called a ***total order*** or a ***linear order***. A totally ordered set is also called a ***chain***.

Hasse Diagrams

If (S,R) is a partially ordered set, we can represent this group in a schematic format called (Hasse diagram), where S is a finite set, as follows:

We represent each element of S in a small circle and if $a < b$ we place b above a and connect between them by a straight line segment, ignoring the cut of the lines we automatically get by means of a transitive property.
For example, if $(a < b) \wedge (b < c)$ and there is no x such that $a < x < b$,also there is no y such that $b < y < c$, then we get a straight line segment between a and b and between b and c but we do not get between a and c.

Example 1. Let R be a relation defined on the set $\mathbb{Z}^+ : a,b \in \mathbb{Z}^+ \ , \ a\ R\ b \Leftrightarrow a \mid b$

(i) Show that R is a partial ordering relation (poset) on $\mathbb{Z}^+$.

(ii) In case R is defined on $\mathbb{Z}$, is R still a partial ordering relation on $\mathbb{Z}$, why?

(iii) Draw the Hasse diagram representing the partial ordering relation R on the set $A = \{1,2,3,4,6,8,12,24\}$

(iv) Decide whether R is totally ordering relation on $\mathbb{Z}^+$, why?

(v) In case R is defined on $B = \{1,3,9,27,81\}$, is R a total ordering relation on B, why?

(vi) Draw the Hasse diagram representing the totally ordering relation R on the set $B = \{1,3,9,27,81\}$

Solution: (i)

1- $\forall a \in \mathbb{Z}^+ \ , \ a|a \ \Rightarrow \therefore a\,R\,a \ \Rightarrow \ R \text{ is } \textit{reflexive}$.

2- $a, b \in \mathbb{Z}^+ \ , \ a\,R\,b \Rightarrow a|b \Rightarrow b = m_1 a : m_1 \in \mathbb{Z}^+$

$b\,R\,a \Rightarrow b|a \Rightarrow a = m_2 b : m_2 \in \mathbb{Z}^+$

$(\times) \quad \Rightarrow$ ______________

$ab = m_1 m_2\, ab$

$\div ab \Rightarrow 1 = m_1 m_2 \Rightarrow \therefore \ m_1 = m_2 = 1 \ \Rightarrow \therefore \ a = b \Rightarrow \therefore \ R \text{ is } \textit{antisymmetric}$

3- $a, b, c \in \mathbb{Z}^+ \ , \ a\,R\,b \Rightarrow a|b \Rightarrow b = m_1 a : m_1 \in \mathbb{Z}^+$

$b\,R\,c \Rightarrow b|c \Rightarrow c = m_2 b : m_2 \in \mathbb{Z}^+$

$(\times) \quad \Rightarrow$ ______________

$bc = m_1 m_2\, ab$

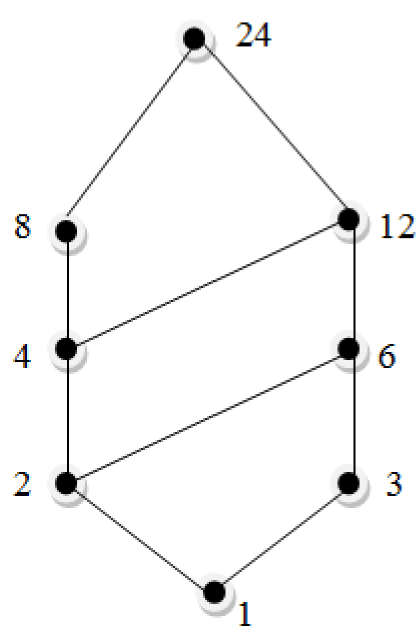
$\Rightarrow \therefore \quad R \text{ is } \textit{transitive}$

$\because$ R is reflexive , *antisymmetric* and *transitive* $\therefore$ R is partial ordering .

(ii) $\because 3, -3 \in \mathbb{Z} \ , \ : 3|(-3) \ \wedge \ -3|3 \ \ \textit{but} \ \ 3 \neq -3 \ \Rightarrow \ R$ is not *antisymmetric*

$\therefore$ R is not a *partial ordering* on $\mathbb{Z}$.

(iii) $R = \left\{ \begin{array}{c} (1,1), (2,2), (3,3), (4,4), (6,6), (8,8), (12,12), (24,24), \\ (1,2), (1,3), (1,4), (1,6), (1,8), (1,12), (1,24), \\ (2,4), (2,6), (2,8), (2,12), (2,24) \\ (3,6), (3,12), (3,24), (4,8), (4,12), (4,12) \\ (6,12), (6,24), (8,24), (12,24) \end{array} \right\}$



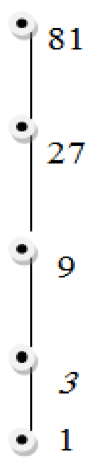
Hasse diagram of R

(iv) $3\,,5 \in \mathbb{Z}^+$, $\because 3 \nmid 5 \;\wedge\; 5 \nmid 3 \;\Rightarrow\; 3\,,5$ $\textit{incomparable} \Rightarrow \therefore R$ is not a total ordering

(v) $\because$ *all elements in B are powers of* $3 \Rightarrow \forall\, a, b \in B\,,\; a|b$ or $b|a$

(B, R) *is comparable* $\Rightarrow \therefore (B, R)$ is ***totally ordered.***

(iv) $R = \left\{ \begin{matrix} (1,1), (3,3), (9,9), (27,27), (81,81), (1,3), (1,9), (1,27), (1,81) \\ (3,9), (3,27), (3,81), (9,27), (9,81), (27,81) \end{matrix} \right\}$



Hasse diagram of R (chain)

Example 2. Let R be a relation defined on the set $\mathbb{Q}^+$:

$$a, b \in \mathbb{Q}^+ \quad , \ a\ R\ b \ \Leftrightarrow \ \frac{a}{b} \in \mathbb{Z}^+$$

(i) Show that R is a partial ordering relation on $\mathbb{Q}^+$.

(ii) Decide whether R is totally ordering relation on $\mathbb{Q}^+$, why?

(iii) Draw the Hasse diagram representing the partial ordering relation R on the set

$$A = \{\frac{1}{6}, \frac{1}{3}, \frac{1}{2}, 1,2,3,6\}$$

$Solution$: (i)

(1) $\forall a \in \mathbb{Q}^+ , \frac{a}{a} = 1 \in \mathbb{Z}^+ \Rightarrow \therefore \ a\ R\ a \Rightarrow \therefore \quad R$ is $reflexive$

(2) $a, b \in \mathbb{Q}^+ \quad , \ a\ R\ b \ \Leftrightarrow \frac{a}{b} = m_1 \in \mathbb{Z}^+ \ \& \ b\ R\ a \ \Leftrightarrow \frac{b}{a} = m_2 \in \mathbb{Z}^+$

$(\times) \Rightarrow \frac{a}{b} \times \frac{b}{a} = m_1 m_2 \Rightarrow 1 = m_1 m_2 \Rightarrow m_1 = m_2 = 1 \Rightarrow \therefore a = b$

$\therefore \ R \ is \ antisymmetric$

(3) $a, b, c \in \mathbb{Q}^+, a\ R\ b \ \Leftrightarrow \frac{a}{b} = m_1 \in \mathbb{Z}^+ \ \& \ b\ R\ c \ \Leftrightarrow \frac{b}{c} = m_2 \in \mathbb{Z}^+$

$(\times) \Rightarrow \frac{a}{b} \times \frac{b}{c} = m_1 m_2 \Rightarrow \frac{a}{c} = m_1 m_2 = m \Rightarrow \therefore aRc : m = m_1 m_2 \in \mathbb{Z}^+$

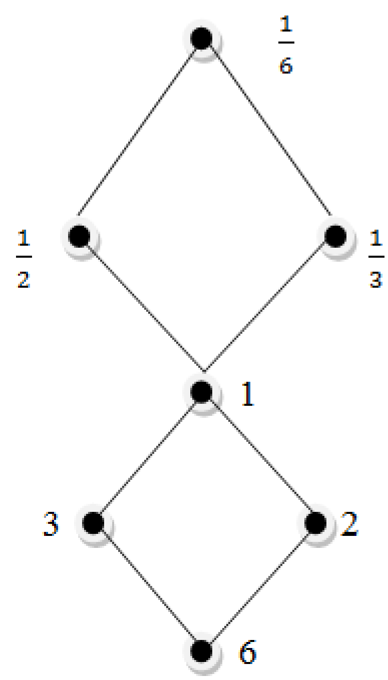
$\therefore \ R \ is \ transitive$

$\because \ R \ is \ reflexive$, $antisymmetric$ and $transitive$

$\therefore \ R$ is partial ordering relation .

(ii) $3 , 5 \in \mathbb{Q}^+ , \frac{3}{5} \notin \mathbb{Z}^+ \wedge \frac{5}{3} \notin \mathbb{Z}^+ \Rightarrow 3 , 5$ *incomparable* $\Rightarrow \therefore$ *R is not totally ordering relation .*

(iii)

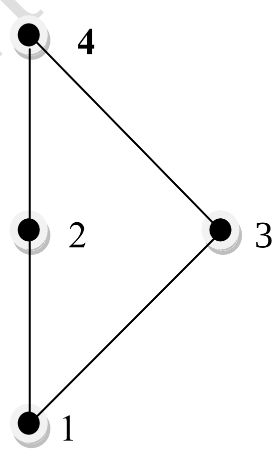


Hasse diagram of R

Example 3. Draw the Hasse diagram representing the partial ordering relation

$$R = \{(1,1), (2,2), (3,3), (4,4), (1,2), (1,3), (1,4), (2,4), (3,4)\}$$

on the set $A = \{1,2,3,4\}$

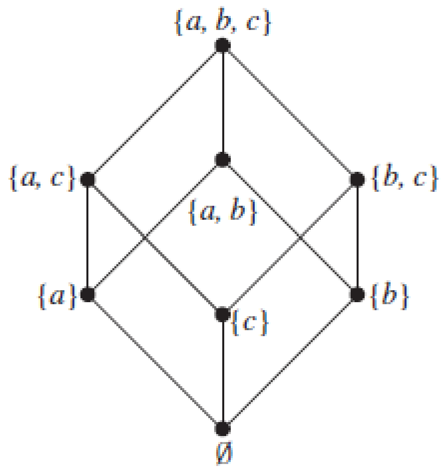


Example 4. Let $\leq$ be a partial ordering relation defined on the set $\mathcal{P}(U)$ where $U = \{a, b, c\}$:

$$A \leq B \Leftrightarrow A \subseteq B$$

Draw the Hasse diagram representing the partial ordering $\leq$ on the (U) .

Solution : $\mathcal{P}(U) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$



The Hasse Diagram
of (P ({a, b, c}),⊆

Exercises

1. Let R be a relation defined on the set $A = \{1,2,3,4,5,6,7,8,9,10\}$

$$a, b \in A \quad , \quad a\ R\ b \Leftrightarrow a \mid b$$

(i) Show that R is a partial ordering relation on .

(ii) Decide whether R is totally ordering relation on A, why?

(iii) Draw the Hasse diagram representing the partial ordering relation R on the set A

Solution :

2. Let R be a relation defined on the set $\mathbb{N} = \{1,2,3,\dots\}$:

$$a, b \in \mathbb{N}, \quad a\, R\, b \Leftrightarrow \frac{b}{a} = 2^k \;: k \in \{0,1,2,\dots\}$$

(i) Show that R is a partial ordering relation on $\mathbb{N}$.

(ii) Decide whether R is totally ordering relation on $\mathbb{N}$, why?

(iii) Draw the Hasse diagram representing the partial ordering relation R on the set $A = \{1,2,3,\dots,12\}$.

Solution : (i)

1- $\forall\, a \in \mathbb{N}, \because \frac{a}{a} = 1 = 2^0 \Rightarrow \therefore a\, R\, a \Rightarrow \therefore \quad R \text{ is } \textit{reflexive}$

2- $a, b \in \mathbb{N} \;,\; a\, R\, b \Leftrightarrow \frac{b}{a} = 2^{k_1}$

$\wedge$

$b\, R\, a \Leftrightarrow \frac{a}{b} = 2^{k_2} \;: k_1, k_2 \in \{0,1,2,\dots\}$

$(\times) \quad \Rightarrow$ ______________

$$\frac{b}{a} \times \frac{a}{b} = 1 = 2^{k_1+k_2}$$

$$1 = 2^0 = 2^{k_1+k_2} \Rightarrow k_1 + k_2 = 0 \Rightarrow k_1 = k_2 = 0$$

3- $a, b, c \in \mathbb{N} \;,\; a\, R\, b \Leftrightarrow \frac{b}{a} = 2^{k_1}$

$\wedge$

$b\, R\, c \Leftrightarrow \frac{c}{b} = 2^{k_2} \quad : k_1, k_2 \in \{0,1,2,\dots\}$

$(\times) \quad \Rightarrow$ ______________

$$\frac{b}{a} \times \frac{c}{b} = \frac{c}{a} = 2^{k_1+k_2} = 2^k \qquad : k_1 + k_2 = k \in \{0,1,2,\dots\}$$

$$\Rightarrow \therefore \; a\, R\, c \Rightarrow \therefore \quad R \text{ is } \textit{transitive}$$

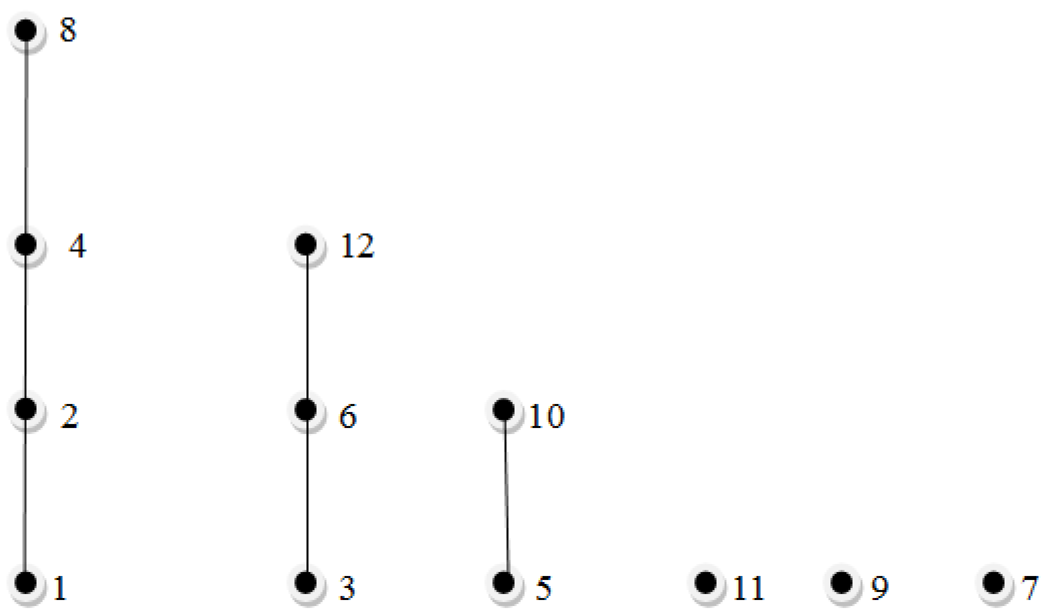
$\because \quad R$ is *reflexive* , *antisymmetric* and *transitive*

$\therefore \quad R$ *is partial ordering relation .*

(ii) $3, 4 \in \mathbb{N} \quad \frac{3}{4} \neq 2^k \;\wedge\; \frac{4}{3} \neq 2^k \;: k \in \{0,1,2,\dots\}$

$\Rightarrow 3, 4$ *incomparable* $\Rightarrow \therefore$ R *is not totally ordering relation .*

(iv) $R = \left\{ \begin{matrix} (1,1),(2,2),(3,3),\dots,(12,12),(1,2),(1,4),(1,8),(2,4),(2,8), \\ (3,6),(3,12),(4,8),(5,10),(6,12) \end{matrix} \right\}$



Hasse diagram of R

3. Let T be a relation defined on the set $A = \{1,2,3,6\}$:

$$x, y \in A \ , \ x \, T \, y \Leftrightarrow \frac{x}{y} \ \ is \ odd$$

(i) List all ordered pairs of T.

(ii) Represent the relation T by diagram .

(iii) Show that T is a partial ordering relation on A .

(iv) Decide whether T is totally ordering relation on A, why?

(v) Draw the Hasse diagram representing the partial ordering relation T on the set A .

4. Let T be a relation defined on the set $\mathbb{Z}$:

$$x, y \in \mathbb{Z} \;, \quad x\,T\,y \;\Leftrightarrow\; x - y = 2k \qquad : k \in \{0,1,2,\dots\}$$

(i) Show that T is a partial ordering relation on $\mathbb{Z}$.

(ii) Decide whether T is totally ordering relation on $\mathbb{Z}$, why?

(iii) Draw the Hasse diagram representing the partial ordering relation T on the set $A = \{0,1,2,3\}$.

Solution: (i)

1- $\forall x \in \mathbb{Z} \;, \; x - x = 0 = 2(0), \; is\ even \;\Rightarrow \therefore\; x\,T\,x \;\Rightarrow\; T \;\; is\;\; reflexive$

2- $x, y \in \mathbb{Z} \;, \quad x\,T\,y \;\Leftrightarrow\; x - y = 2k_1 \quad : k_1 \in \{0,1,2,\dots\}$

$\wedge$

$y\,T\,x \;\Leftrightarrow\; y - x = 2k_2 \quad : k_2 \in \{0,1,2,\dots\}$

$(+) \quad \Rightarrow$ ______________

$0 = 2(k_1 + k_2) \;\Rightarrow\; k_1 = k_2 = 0$

$\therefore\; x - y = 0 \;\Rightarrow x = y \;\Rightarrow\; T \;\; is\ antisymmetric$

3- $x, y, z \in \mathbb{Z} \;, \quad x\,T\,y \;\Leftrightarrow\; x - y = 2k_1 \quad : k_1 \in \{0,1,2,\dots\}$

$\wedge$

$y\,T\,z \;\Leftrightarrow\; y - z = 2k_2 \quad : k_2 \in \{0,1,2,\dots\}$

$(+) \quad \Rightarrow$ ______________

$x - z = 2(k_1 + k_2) = \; 2k \;\; \Rightarrow x\,T\,z$

$: k_1 + k_2 = k \in \{0,1,2,\dots\} \qquad \Rightarrow \;\; T\ is \;\; transitive$

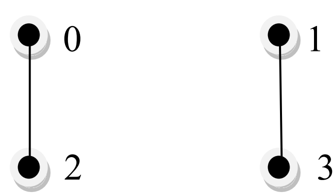
$\because\;\; T$ is $reflexive$, $antisymmetric$ and $transitive$

$\therefore\;\; T$ *is partial ordering relation .*

(ii) $2\,,5 \in \mathbb{Z} \;, \quad 5 - 2 = 3 \quad (is\ odd) \;\; \wedge \;\; 2 - 5 = \; -3 \;\; (is\ odd)$

$2\,,5$ $incomparable \;\Rightarrow\; \therefore\;\; T$ *is not totally ordering relation .*

(iii) $T = \{(0,0), (1,1), (2,2), (3,3), (2,0), (3,1)\}$



Hasse diagram of T

5. Let T be a relation defined on the set $\mathbb{Z}^* = \{\dots, -2, -1, 1, 2, \dots\}$:

$$a, b \in \mathbb{Z}^*, a\ T\ b \Leftrightarrow \frac{a}{b} = 3^k \quad : k \in \{0, 1, 2, \dots\}$$

(i) Show that T is a partial ordering relation on $\mathbb{Z}^*$.

(ii) Decide whether T is totally ordering relation on $\mathbb{Z}^*$, why?

(iii) Draw the Hasse diagram representing the partial ordering relation T on the set $A = \{-27, -18, -9, -6, -3, 1, 2, 3, 6, 9\}$.

Solution : (i)

6. Let T be a relation defined on the set $\mathbb{N} = \{1,2,3,\dots\}$:

$$x, y \in \mathbb{N}, \quad x\,T\,y \Leftrightarrow x = y^k \quad : k \in \{0,1,2,\dots\}$$

(i) Show that T is a partial ordering relation on $\mathbb{N}$.

(ii) Decide whether T is totally ordering relation on $\mathbb{N}$, why?

(iii) Draw the Hasse diagram representing the partial ordering relation T on the set $A = \{1,2,3,4\}$.

Solution : (i)

1- $\forall x \in \mathbb{N}\ ,\ x = x^1 \Rightarrow \therefore\ x\,T\,x \Rightarrow \therefore\ T$ *is reflexive*

2- $x, y \in \mathbb{N}, \quad x\,T\,y \Leftrightarrow x = y^{k_1} \quad : k_1 \in \{0,1,2,\dots\}$

$\wedge$

$y\,T\,x \Leftrightarrow y = x^{k_2} \quad : k_2 \in \{0,1,2,\dots\}$

(*by substitution*) $\Rightarrow x = x^{k_1 k_2} \Rightarrow k_1 k_2 = 1 \Rightarrow k_1 = k_2 = 1$

$\Rightarrow x = y \Rightarrow \therefore$ T *is antisymmetric*

3- $x, y, z \in \mathbb{N}, \quad x\,T\,y \Leftrightarrow x = y^{k_1} \quad : k_1 \in \{0,1,2,\dots\}$

$\wedge$

$y\,T\,z \Leftrightarrow y = z^{k_2} \quad : k_2 \in \{0,1,2,\dots\}$

(*by substitution*) $\Rightarrow x = z^{k_1 k_2} \Rightarrow x = z^k \quad : k_1 k_2 = k \in \{0,1,2,\dots\}$

$\Rightarrow x\,T\,z \Rightarrow \therefore$ T *is transitive*

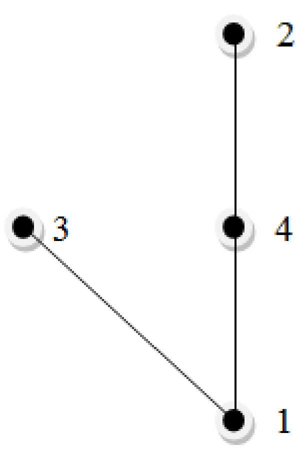
$\because$ T is *reflexive* , *antisymmetric* and *transitive*

$\therefore$ T *is partial ordering relation on* $\mathbb{N}$.

(ii) $2\ ,\ 5 \in \mathbb{N}\ ,\ 2 \neq 5^k \quad \wedge \quad 5 \neq 2^k$

$\therefore 2\ ,5$ *incomparable* $\Rightarrow \therefore$ T *is not totally ordering relation* .

(iii) $T = \{(1,1), \dots, (4,4), (1,2), (1,3), (1,4)(4,2)\}$



Hasse diagram of T

7. Let $T = \{(x,x),(x,z),(y,x),(y,y),(y,z),(z,z)\}$ be a relation defined on the set $B = \{x,y,z\}$

(i) Represent the relation T by diagram .

(ii) Show that T is a partial ordering relation on B .

(iii) Decide whether T is totally ordering relation on A, why?

(iv) Draw the Hasse diagram representing the partial ordering relation T on the set B .

8. Let S be a relation defined on the set $\mathbb{N} = \{1,2,3,\dots\}$:

$$x, y \in \mathbb{N} \ , \ x\,S\,y \Leftrightarrow \frac{x}{y} \ \ is\ odd$$

(i) Show that S is a partial ordering relation on $\mathbb{N}$.

(ii) Decide whether S is totally ordering relation on $\mathbb{N}$, why?

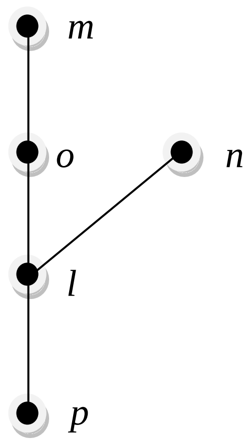
(iii) Draw the Hasse diagram representing the partial ordering relation S on the set $A = \{1,2,6\}$.

Solution: (i)

9. Let T be a partial ordering relation defined on the set $C = \{l, m, n, o, p\}$ shown in the given Hasse diagram

(i) List all ordered pairs of T.

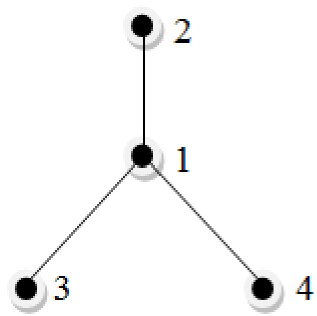
(ii) Decide whether T is totally ordering relation on C, why?



10. Let S be a partial ordering relation defined on the set $A = \{1,2,3,4\}$ shown in the given Hasse diagram

(i) List all ordered pairs of S.

(ii) Decide whether S is totally ordering relation on A, why?



Solution:

$$S = \{(1,1),(2,2),(3,3),(4,4),(3,1),(4,1),(1,2),(3,2),(4,2)\}$$

$\because$ $(3,4) \wedge (4,3) \notin S$ $\therefore$ 3 , 4 *incomparable* $\Rightarrow$ $\therefore$ S *is not totally ordering relation* .

11. Let S be a relation defined on the set $\mathbb{N} = \{1,2,3,\dots\}$:

$$x, y \in \mathbb{N} \ , \ x \, S \, y \Leftrightarrow \frac{x}{y} \ \ is\ odd$$

(i) Show that S is a partial ordering relation on $\mathbb{N}$.

(ii) Decide whether S is totally ordering relation on $\mathbb{N}$, why?

(iii) Draw the Hasse diagram representing the partial ordering relation S on the set $A = \{1,2,3,4,5,6,9,10,12\}$.

Solution: (i)

12. Let T be a relation defined on the set $\mathbb{Z}^* = \{\dots,-2,-1,1,2,\dots\} = \mathbb{Z} - \{0\}$:

$$a,b \in \mathbb{Z}^*, a\,T\,b \Leftrightarrow \frac{a}{b} = 2^m\,3^n \;:\; m,n \in \{0,1,2,\dots\}$$

(i) Show that T is a partial ordering relation on $\mathbb{Z}^*$.

(ii) Decide whether T is totally ordering relation on $\mathbb{Z}^*$, why?

(iii) Draw the Hasse diagram representing the partial ordering relation T on the set $A = \{1,2,3,5,6\}$.

Solution : (i)

1- $\forall a \in \mathbb{Z}^*, \frac{a}{a} = 1 = 2^0 3^0 \Rightarrow \therefore a\,T\,a \Rightarrow \therefore \quad T$ *is reflixive*

2- $a,b \in \mathbb{Z}^*, a\,T\,b \Leftrightarrow \frac{a}{b} = 2^{m_1}\,3^{n_1} \;:\; m_1,n_1 \in \{0,1,2,\dots\}$

$\wedge$

$a\,T\,b \Leftrightarrow \frac{b}{a} = 2^{m_2}\,3^{n_2} \;:\; m_2,n_2 \in \{0,1,2,\dots\}$

$(\times) \quad \Rightarrow$ ______________

$$\frac{a}{b}\,\frac{b}{a} = 1 = 2^{m_1}\,3^{n_1}\,2^{m_2}\,3^{n_2} = 2^{m_1+m_2}\;3^{n_1+n_2} \Rightarrow m_1+m_2 = 0 \wedge n_1+n_2 = 0$$

$$\frac{a}{b} = 2^0 3^0 = 1 \Rightarrow \therefore \; a = b \Rightarrow \therefore \quad T \text{ is } \textit{antisymmetric}$$

3- $a,b,c \in \mathbb{Z}^*$, $\quad a\,T\,b \Leftrightarrow \frac{a}{b} = 2^{m_1}\,3^{n_1} \;:\; m_1,n_1 \in \{0,1,2,\dots\}$

$\wedge$

$b\,T\,c \Leftrightarrow \frac{b}{c} = 2^{m_2}\,3^{n_2} \;:\; m_2,n_2 \in \{0,1,2,\dots\}$

$(\times) \quad \Rightarrow$ ______________

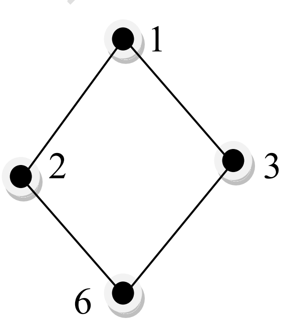
$$\frac{a}{b} \times \frac{b}{c} = \frac{a}{c} = 2^{m_1}\,3^{n_1} \times 2^{m_2}\,3^{n_2} = 2^{m_1+m_2}\,3^{n_1+n_2} = 2^m 3^n$$

$$: m_1+m_2 = m \wedge n_1+n_2 = n \in \{0,1,2,\dots\}$$

$$\frac{a}{c} = 2^m 3^n \Rightarrow \therefore \; a\,T\,c \Rightarrow \therefore \quad T \text{ is } \textit{transitive}$$

$\because \; T$ is *reflexive* , *antisymmetric* and *transitive*

$\therefore \; T$ *is partial ordering relation on* $\mathbb{N}$.



(ii) $2 \;,\; 3 \in \mathbb{Z}^*$, $\frac{2}{3} \neq 2^m 3^n \;\wedge\; \frac{3}{2} \neq 2^m 3^n : m,n \in \{0,1,2,\dots\}$

$\therefore$ 2 , 3 *incomparable* $\Rightarrow \therefore \; T$ *is not totally ordering relation .*

(iii) $T = \{(1,1),\dots,(6,6),(2,1),(3,1),(6,1),(6,2),(6,3)\}$

13. Let T be a relation defined on the set $\mathbb{Z}^* = \{\dots, -2, -1, 1, 2, \dots\} = \mathbb{Z} - \{0\}$

$$x, y \in \mathbb{Z}^*, \quad x\, R\, y \Leftrightarrow x = y^{2k+1} \quad : k \in \{0, 1, 2, \dots\}$$

(i) Show that T is a partial ordering relation on $\mathbb{Z}^*$.

(ii) Decide whether T is totally ordering relation on $\mathbb{Z}^*$, why?

(iii) Draw the Hasse diagram representing the partial ordering relation T on the set $A = \{1,2,3,4,8,27\}$.

Solution : (i)

:

14. Let R be a relation defined on the set $\mathbb{Z}^+$: $a, b \in \mathbb{Z}^+$, $a\,R\,b \Leftrightarrow a^2 \mid b^2$

(i) Show that R is a partial ordering relation (poset) on $\mathbb{Z}^+$.

(ii) In case R is defined on $\mathbb{Z}$, decide whether R a partial ordering relation on $\mathbb{Z}$, why?

(iii) Draw the Hasse diagram representing the partial ordering relation R on the set $A = \{1,2,3,4,6,8,12,27\}$

(iv) Decide whether R is totally ordering relation on $\mathbb{Z}^+$, why?

Solution: (i)

1- $\forall a \in \mathbb{Z}^+ ,\ a^2 \mid a^2 \Rightarrow a\,R\,a \Rightarrow R$ *is reflexive*

2- $a, b \in \mathbb{Z}^+ ,\ a\,R\,b \Leftrightarrow a^2 \mid b^2 \Rightarrow b^2 = ma^2 \quad : m \in \mathbb{Z}^+$

$\wedge$

$b\,R\,a \Leftrightarrow b^2 \mid a^2 \Rightarrow a^2 = nb^2 \quad : n \in \mathbb{Z}^+$

$b^2 = mnb^2 \Rightarrow mn = 1 \Rightarrow m = n = 1$

$a^2 = b^2 \overset{\sqrt{\ }}{\Longrightarrow} a = b \Rightarrow R$ *is antisymmetric*

3- $a, b, c \in \mathbb{Z}^+ ,\ a\,R\,b \Leftrightarrow a^2 \mid b^2 \Rightarrow b^2 = ma^2 : m \in \mathbb{Z}^+$

$\wedge$

$b\,R\,c \Leftrightarrow b^2 \mid c^2 \Rightarrow c^2 = nb^2 : n \in \mathbb{Z}^+$

$c^2 = mna^2 \Rightarrow a^2 \mid c^2 \Rightarrow a\,R\,c \Rightarrow R$ *is transitive*

$\because$ R is *reflexive* , *antisymmetric* and *transitive*

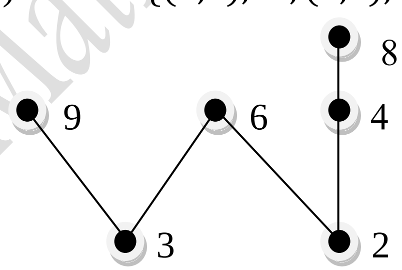
$\therefore$ R *is partial ordering relation on* $\mathbb{Z}^+$.

(ii) $-2 , 2 \in \mathbb{Z}$

$-2\,R\,2 :\ (-2)^2 | 2^2 \wedge -2\,R\,2 :\ 2^2 | (-2)^2$ but $-2 \neq 2 \Rightarrow$ *R is not antisymmetric*

$\therefore$ R is *not partial ordering relation on* $\mathbb{Z}$

(iii) $R = \{(2,2), \dots, (9,9), (2,4), (2,6), (2,8), (3,6), (3,9), (4,8)\}$



$2 , 3 \in \mathbb{Z}^+ , 2^2 \nmid 3^2 \wedge 3^2 \nmid 2^2 \Rightarrow \therefore 2 , 3$ *incomparable*

15. Let $T = \{(a,a),(a,b),(b,b),(c,c)\}$ be a relation defined on the set $A = \{a,b,c\}$. Decide whether T is reflexive , symmetric , antisymmetric , transitive , equivalence , partial ordering relation . Why?

Solution:

1- $\because\ (a,a),(b,b),(c,c) \in T \ \Rightarrow \ T \ \ is\ reflexive$

2- $\because (a,b) \in T \ \wedge \ (b,a) \notin T \ \Rightarrow \ T \ is\ not\ symmetric$

3- $\because (a,b) \in T \ \wedge \ (b,a) \notin T \ \Rightarrow \ T \ is\ antisymmetric$

4- $\because (a,a) \in T \ \wedge \ (a,b) \in T \ \Rightarrow (a,b) \in T$

& $\because (a,b) \in T \ \wedge \ (b,b) \in T \ \Rightarrow (a,b) \in T \ \Rightarrow \ T \ \ is\ transitive$

$\because T \ \ is\ reflexive \ ,antisymmetric \ \ and\ transitive$

$\Rightarrow \therefore T \ \ is\ partial\ ordering\ relation\ .$

#

16. Let $R = \{(a,a),(b,b),(c,c),(d,d)\}$ be a relation defined on the set $A = \{a,b,c,d\}$. Decide whether R is reflexive , symmetric , antisymmetric , transitive , equivalence , partial ordering , totally ordering relation . Why?

Solution:

1- $\because (a,a),(b,b),(c,c),(d,d) \in R \Rightarrow \therefore R\ \ is\ reflexive$

2- $\because (a,a) \wedge (a,a) \in R\ \&\ (b,b) \wedge (b,b) \in R$

$\&\ (c,c) \wedge (c,c) \in R\ \&\ (d,d) \wedge (d,d) \in R \Rightarrow \therefore R\ is\ symmetric$

3- $\because (a,a) \wedge (a,a) \in R \Rightarrow \therefore a = a$, also same for $(b,b),(c,c),(d,d)$

$\therefore R\ is\ antisymmetric$

4- $\because (a,a) \wedge (a,a) \in R \Rightarrow (a,a) \in R$, also same for $(b,b),(c,c),(d,d)$

$\therefore R\ is\ transitive$

5- $\because R\ \ is\ reflexive\ ,symmetric\ and\ transitive$

$\Rightarrow \therefore R\ \ is\ equivalence\ relation$

6- $\because R\ \ is\ reflexive\ ,antisymmetric\ and\ transitive$

$\Rightarrow \therefore R\ \ is\ partial\ ordering\ relation$

7- $\because (a,b) \wedge (b,a) \notin R \Rightarrow a\ and\ b\ incomparable$

$\Rightarrow \therefore R\ \ is\ not\ totally\ ordering\ relation$

$Finally\ R\ \ is\ equivalence\ relation\ \ \&\ \ partial\ ordering\ relation$.

#

17. Let $R = \{(x, x)\}$ be a relation defined on the set $A = \{x\}$.
Decide whether R is reflexive , symmetric , antisymmetric , transitive , equivalence , partial ordering , totally ordering relation . Why?

18. Let R be a relation defined on the set $\mathbb{Z}^+ = \{1,2,3,\dots\}$

$$m,n \in \mathbb{Z}^+ \quad , \qquad m\,R\,n \Leftrightarrow m+n=20$$

Decide whether R is reflexive , symmetric , antisymmetric , transitive , equivalence , partial ordering relation . Why?

Solution :

1- $\because\ 5+5 \neq 20 \Rightarrow (5,5) \notin R \Rightarrow \therefore\ R\ \ is\ not\ reflexive$

2- $m\,,n \in \mathbb{Z}^+\ ,\ m\,R\,n \Leftrightarrow m+n=20$

$\xRightarrow{\text{(commutative)}} n+m=20 \Rightarrow \therefore\ n\,R\,m \Rightarrow \therefore\ R\ is\ symmetric$

3- $\because\ 7\,R\,13\ :7+13=20 \quad \wedge \quad 13\,R\,7\ :13+7=20$

but $7 \neq 13 \Rightarrow \therefore\ R\ \ is\ not\ antisymmetric.$

4- $\because\ 8\,R\,12\ :8+12=20 \quad \wedge \quad 12\,R\,8\ :12+7=20$

But $(8,8) \notin R\ :8+8=16 \neq 20 \Rightarrow \therefore\ R\ \ is\ not\ transitive.$

Finally , $R\ is\ only\ symmetric$.

#

19. Let $T = \{(a,a),(a,c),(b,b),(b,c),(c,a),(d,d)\}$ be a relation defined on the set $A = \{a,b,c,d\}$. Decide whether T is reflexive , symmetric , antisymmetric ,transitive . Why?

20. Let R be a relation defined on the set $A = \{0,1,2,3\}$

$$a,b \in A \quad , \quad a\,R\,b \Leftrightarrow a \leq 2b$$

(i) List all the ordered pairs of R.

(ii) Represent R in a diagram .

(iii) Decide whether R is reflexive , symmetric , antisymmetric , transitive . Why?

21. Let R be a relation defined on the set $\mathbb{Z}^{+} = \{1,2,3, \dots\}$

$$m, n \in \mathbb{Z}^{+} \quad , \quad m\, R\, n \Leftrightarrow 6 | m\, n$$

Decide whether R is reflexive , symmetric , antisymmetric , transitive . Why?

22. Suppose T is a relation defined on the integers set $\mathbb{Z}$

$$m, n \in \mathbb{Z}, \quad m\,T\,n \iff m + n \geq 2$$

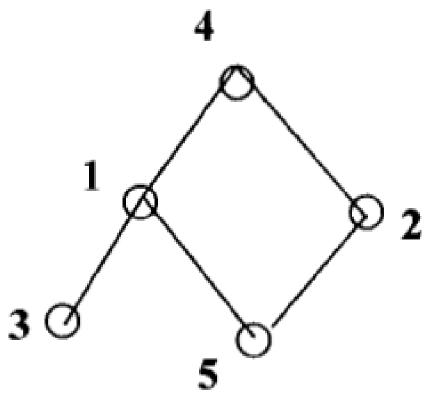
Decide whether the relation T is *reflexive*, *symmetric*, *antisymmetric*, and/or *transitive*.

Solution:

23. Let T be a partial ordering relation defined on the set $A = \{1,2,3,4,5\}$ shown in the given Hasse diagram

(i) List all ordered pairs of T .

(ii) Decide whether T is totally ordering relation on A, why?



24. Let R be a relation defined on the set $\mathbb{Z}^+ = \{1,2,3,\dots\}$:

$$m,n \in \mathbb{Z}^+ , \quad m\,R\,n \iff m = n^a \quad : a \in \{0,1,2,\dots\}$$

(i) Show that R is a partial ordering relation on $\mathbb{Z}^+$.

(ii) Decide whether R is totally ordering relation on $\mathbb{Z}^+$, why?

(iii) Draw the Hasse diagram representing the partial ordering relation R on the set $A = \{2,3,4,5,6,7,8,9\}$

25. Let T be a relation defined on the set $\mathbb{N} = \{1,2,3,...\} : m\ T\ n \Leftrightarrow m < n$
Decide whether the relation T is *reflexive*, *symmetric*, *antisymmetric*, and/or *transitive*.

Solution: