Lebesgue Integral

Mongi BLEL

King Saud University

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Measurable Functions

General Properties of Measurable Functions

Let X and Y be two nonempty sets. We showed in the previous chapter that the pull back of a σ-algebra under a mapping $f\colon X \longrightarrow Y$ is a σ-algebra of X.

Definition

If $(X, \mathscr{A})$ and $(Y, \mathscr{B})$ are two measurable spaces. A mapping $f\colon X \longrightarrow Y$ is called measurable if $f^{-1}(\mathscr{B}) \subset \mathscr{A}$.

Theorem

Let $(X, \mathscr{A})$ and $(Y, \mathscr{B})$ be two measurable spaces, and suppose that $\mathcal{B}$ generates the σ-algebra $\mathscr{B}$. A function $f : X \to Y$ is measurable if and only if $f^{-1}(\mathcal{B}) \subset \mathscr{A}$.

Proof

The sufficient condition is just the definition of the measurability. For the "if" direction, define

$$\mathcal{H} = \{V \in \mathscr{B} \colon f^{-1}(V) \in \mathscr{A}\}.$$

$\mathcal{H}$ is a σ-algebra since the operation of taking the inverse image commutes with the set operations of union, intersection and complement.

If $\mathcal{B} \subset \mathcal{H}$, therefore, $\sigma(\mathcal{B}) \subset \sigma(\mathcal{H})$. But $\mathscr{B} = \sigma(\mathcal{B})$ and $\mathcal{H} = \sigma(\mathcal{H})$ since $\mathcal{H}$ is a σ-algebra. This means that $f^{-1}(V) \in \mathscr{A}$ for every $V \in \mathscr{B}$. $\square$

Remark

To show that a mapping $f\colon X \longrightarrow Y$ is measurable; it suffices to give a set $\mathcal{C}$ which generates $\mathscr{B}$ and $f^{-1}(\mathcal{C}) \subset \mathscr{A}$.

Proposition

Let $(X, \mathscr{A})$ be a measurable space and let $f\colon X \longrightarrow \mathbb{R}$ (or in $\overline{\mathbb{R}}$) be a function. The function f is measurable, if one of the following conditions is fulfilled

1. $\forall a \in \mathbb{R}\ \{x \in X;\ f(x) \geq a\} \in \mathscr{A}$.
2. $\forall a \in \mathbb{R}\ \{x \in X;\ f(x) < a\} \in \mathscr{A}$.
3. $\forall a \in \mathbb{R}\ \{x \in X;\ f(x) \leq a\} \in \mathscr{A}$.
4. $\forall a, b \in \mathbb{R}\ \{x \in X;\ a < f(x) < b\} \in \mathscr{A}$.
5. $\forall a, b \in \mathbb{R}\ \{x \in X;\ a \leq f(x) < b\} \in \mathscr{A}$.

The space $\mathbb{R}$ (resp $\overline{\mathbb{R}}$) is endowed with the Borel σ-algebra $\mathscr{B}_{\mathbb{R}}$ (resp $\mathscr{B}_{\overline{\mathbb{R}}}$).
This proposition is deduced from the fact that Borel $\sigma-$algebra is generated by any one of the following set of intervals
a) $\{[a, +\infty[;\ a \in \mathbb{R}\}$,
b) $\{]a, +\infty[;\ a \in \mathbb{R}\}$,
c) $\{] - \infty, a[;\ a \in \mathbb{R}\}$,
d) $\{] - \infty, a];\ a \in \mathbb{R}\}$,
e) $\{]a, b[;\ a, b \in \mathbb{R}\}$,
f) $\{[a, b[;\ a, b \in \mathbb{R}\}$,
g) $\{]a, b];\ a, b \in \mathbb{R}\}$,
h) $\{[a, b];\ a, b \in \mathbb{R}\}$.

$\square$

Operations of Measurable Functions

Proposition

Let $(X_0, \mathscr{A}_0)$, $(X_1, \mathscr{A}_1)$ and $(X_2, \mathscr{A}_2)$ three measurable spaces and let $f_1 : X_0 \longrightarrow X_1$ and $f_2 : X_1 \longrightarrow X_2$ be two measurable mappings, then the mapping $f_2 \circ f_1$ is measurable.

The proposition results from the following that

$$(f_2 \circ f_1)^{-1}(\mathscr{A}_2) = f_1^{-1}(f_2^{-1}(\mathscr{A}_2)) \subset f_1^{-1}(\mathscr{A}_1) \subset \mathscr{A}_0.$$

Proposition

Let $(X, \mathscr{A})$ be a measurable space.
a) If $f : X \longrightarrow \overline{\mathbb{R}}$ is measurable of $(X, \mathscr{A})$, then $|f|$ is measurable.
b) If $(f_n)_{n\in\mathbb{N}}$ is a sequence of measurable functions of $(X, \mathscr{A})$ with real values, then the functions g,h,k defined by $g = \sup_{n\in\mathbb{N}} f_n$, $h = \overline{\lim}_{n\to+\infty} f_n$ and $k = \underline{\lim}_{n\to+\infty} f_n$ are measurable.

Proof

a) If $a < 0$; $\{x \in X;\ |f(x)| > a\} = X$.
If $a \geq 0$; $\{x \in X;\ |f(x)| > a\} = \{x \in X;\ f(x) > a\} \cup \{x \in X;\ f(x) < -a\} = f^{-1}(]a, +\infty]) \cup f^{-1}([-\infty, -a[) \in \mathscr{A}$.
b) $\{x \in X;\ g(x) > a\} = \bigcup_{n\in\mathbb{N}}\{x \in X;\ f_n(x) > a\} \in \mathscr{A}$.
$h(x) = \inf_{n\in\mathbb{N}}(\sup_{j\geq n} f_j(x))$

$$\{x \in X;\ h(x) > a\} = \bigcap_{n=1}^{+\infty}\bigcup_{j=n}^{\infty}\{x \in X;\ f_j(x) > a\} \in \mathscr{A}.$$

$k(x) = \sup_{n\in\mathbb{N}}(\inf_{j\geq n} f_j(x))$

$$\{x \in X;\ k(x) > a\} = \bigcup_{n=1}^{+\infty}\bigcap_{j=n}^{\infty}\{x \in X;\ f_j(x) > a\} \in \mathscr{A}.$$

Remark

It results from the previous proposition that if f is measurable then the functions $f^+ = \sup(f, 0)$ and $f^- = \inf(f, 0)$ are measurable, and if $(f_n)_{n\in\mathbb{N}}$ is a sequence of measurable functions which converges point wise toward a function f on X, then f is measurable.

□

Corollary

For any sequence $(f_n)_{n\in\mathbb{N}}$ of real measurable functions on a measurable space X, if $C = \{x \in X;\ \lim\limits_{n\to+\infty} f_n(x) \text{ exists in } \overline{\mathbb{R}}\}$.
Then C is measurable.

Proof

Let $D = C^c$, $D = \{x \in X;\ \underline{\lim}_{n\to+\infty} f_n(x) < \overline{\lim}_{n\to+\infty} f_n(x)\}$. If we set $g = \underline{\lim}_{n\to+\infty} f_n$ and $h = \overline{\lim}_{n\to+\infty} f_n$. For each rational r, let

$$D_r = \{x \in X;\ g(x) < r < h(x)\} = \{g(x) < r\} \cap \{h(x) > r\}$$

which is measurable. $D = \bigcup_{r\in\mathbb{Q}} D_r$ which proves the measurability of D. $\square$

Simple Functions

Definition

Let $(X, \mathscr{A})$ be a measurable space. A function $f : X \longrightarrow \mathbb{R}$ (or $(\overline{\mathbb{R}})$) is called a **simple function** if it is measurable and takes a finite number of values.

Let $f : X \longrightarrow \bar{\mathbb{R}}$ be a simple function. If $\{c_1, \ldots, c_m\}$ is the set of values of f; $c_j \neq c_k$ for $j \neq k$, and $A_j = f^{-1}\{c_j\}$, then $X = \bigcup_{j=1}^m A_j$,

$$A_j \cap A_k = \emptyset \text{ if } j \neq k \text{ and } f = \sum_{j=1}^{m} c_j \chi_{A_j}.$$

We remark that f is measurable if and only if A_j is measurable for all $j = 1, \ldots, m$.

Theorem

Let $(X, \mathscr{A})$ be a measurable space and $f : X \longrightarrow \overline{\mathbb{R}}$

1. If f is a measurable and bounded, there exists a sequence of simple functions which converges uniformly on X to f.
2. If f is a non-negative measurable function. Then there exists a sequence of non-negative simple functions which increases to f.

Proof

1) Let $M > 0$ such that $\forall x \in X$, $|f(x)| < M$. We denote by $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. For $(n,k) \in \mathbb{N}_0 \times \mathbb{Z}$ and $-2^n \leq k \leq 2^n - 1$, we set

$$A_{n,k} = \{x \in X;\ \frac{kM}{2^n} \leq f(x) < \frac{(k+1)M}{2^n}\}$$

and we define f_n by

$$f_n = \sum_{k=-2^n}^{2^n-1} \frac{kM}{2^n} \chi_{A_{n,k}}.$$

The subsets $A_{n,k}$ are measurable and f_n is measurable, for all $n \in \mathbb{N}$.

For any $x_0 \in X$, there exists k_0 such that $x_0 \in A_{n,k_0}$. Then $f_n(x_0) = \frac{Mk_0}{2^n}$ and $|f(x_0) - f_n(x_0)| < \frac{M}{2^n}$. Then the sequence $(f_n)_n$ converges uniformly on X to f.

2) For all $n \in \mathbb{N}$, let $g_n = \inf(f, n) - \frac{1}{n}$. The function g_n is bounded measurable, then from the first case there exists a sequence of simple functions $(f_m)_m$ such that $||f_n - g_n||_\infty < \dfrac{1}{2^n}$. We conclude that

$$\lim_{n\to+\infty} f_n = \lim_{n\to+\infty} g_n = \lim_{n\to+\infty} \inf(f, n) = f.$$

$f_n \leq g_n + \frac{1}{2^n} = \inf(f, n) - \frac{1}{n} + \frac{1}{2^n} \leq \inf(f, n+1) - \frac{1}{n+1} + \frac{1}{2^{n+1}} \leq f_{n+1}$. (It suffices to prove that for n big enough $-\frac{1}{n} + \frac{1}{2^n} < -\frac{1}{n+1} + \frac{1}{2^{n+1}}$.)

□

Integration

For constructing the integral of real measurable functions on a measure space $(X, \mathscr{A}, \mu)$, we proceed by steps. We begin by the case of the integral of simple functions, then we define the integral of non-negative measurable functions by the increasing limit and we show that the monotone limit allows to define the integral of the non-negative measurable functions, and finally the decomposition of a measurable arbitrary functions $f = f^+ - f^-$ as the difference of two non-negative measurable functions extends the definition of the integral to the measurable functions.

Definition

If $f = \sum_{k=1}^{N} \lambda_k \chi_{\{f=\lambda_k\}}$ is a non-negative simple function, we define the integral of f by

$$\int_X f(x) d\,\mu(x) = \sum_{k=1}^{N} \lambda_k \mu(\{f = \lambda_k\})$$

We take the convention that if $A = \{x \in \Omega;\ f(x) = 0\}$ and $\mu(A) = +\infty$, then $\int_X f(x) d\,\mu(x) = 0$. ($0 \times (+\infty) = 0$).
In particular if $f = \chi_A$, where A is a measurable subset, then $\int_X \chi_A(x) d\,\mu(x) = \mu(A)$.

Proposition

Let $\mathscr{E}^+$ be the cone of non-negative simple functions on the measure space $(X, \mathscr{A}, \mu)$. The integral defined on $\mathscr{E}^+$ has the following properties

1. $\forall\ \alpha\ \in \mathbb{R}^+,\quad \forall f \in\ \mathscr{E}^+;\quad \int_X \alpha\ f(x) d\,\mu(x) = \alpha \int_X f(x) d\,\mu(x).$
2. $\forall f, g\ \in \mathscr{E}^+;\quad \int_X (f+g)(x) d\,\mu(x) = \int_X f(x) d\,\mu(x) + \int_X g(x) d\,\mu(x).$

3. $\forall f, g \in \mathscr{E}^+$ such that $f \leq g$; $\displaystyle\int_X f(x) d\mu(x) \leq \int_X g(x) d\mu(x)$.
4. If $(f_n)_n$ is an increasing sequence in $\mathscr{E}^+$ and if $f = \lim\limits_{n \to +\infty} f_n$ is the limit of the sequence $(f_n)_n$ belongs to $\mathscr{E}^+$, then $\displaystyle\int_X f(x) d\mu(x) = \lim_{n \to +\infty} \int_X f_n(x) d\mu(x)$.

Proof

It is evident that if $\alpha \geq 0$ and f and g of $\mathscr{E}^+$ then αf and $f+g \in \mathscr{E}^+$. ($\mathscr{E}^+$ is a convex cone).

1. The first property is evident.
2. Let f and g be two elements of $\mathscr{E}^+$. We denote by F (resp G) the set of values of f (resp of g).

$$f = \sum_{a \in F} a\chi_{\{f=a\}}, \quad g = \sum_{b \in G} b\chi_{\{g=b\}}.$$

We have

$$\forall\ a \in F;\ \{f = a\} = \bigcup_{b \in G} \{f = a, g = b\}.$$

$$\forall\ b \in G;\ \{g = b\} = \bigcup_{a \in F} \{f = a, g = b\}.$$

$$\int_X f(x) d\,\mu(x) = \sum_{a \in F} a\mu\{f = a\} = \sum_{(a,b) \in F \times G} a\mu\{f = a, g = b\}$$

$$\int_X g(x) d\,\mu(x) = \sum_{b \in G} a\mu\{g = b\} = \sum_{(a,b) \in F \times G} b\mu\{f = a, g = b\}$$

$$\int_X f(x) d\,\mu(x) + \int_X g(x) d\,\mu(x) = \sum_{(a,b) \in F \times G} (a+b)\mu\{f = a, g = b\}$$

$\{f + g = u\} = \bigcup_{(a,b) \in F \times G, a+b=u} \{f = a, g = b\}$. It results that

$$\mu\{f+g=u\} = \sum_{(a,b)\in F\times G, a+b=u} \mu\{f=a, g=b\}$$

Then

$$\begin{aligned} \int_X f(x)d\,\mu(x) + \int_X g(x)d\,\mu(x) &= \sum_u u\mu\{f+g=u\} \\ &= \int_X (f+g)(x)d\,\mu(x). \end{aligned}$$

3. If $\int_X f(x)d\,\mu(x) = +\infty$, then $\int_X g(x)d\,\mu(x) = +\infty$. The result is evident if $\int_X f(x)d\,\mu(x) < +\infty$ and $\int_X g(x)d\,\mu(x) = +\infty$.

Assume now that $\int_X f(x)d\,\mu(x) < +\infty$ and $\int_X g(x)d\,\mu(x) < +\infty$, then the subsets $\{x \in X;\ f(x) = +\infty\}$ and $\{x \in X;\ g(x) = +\infty\}$ are null sets.
Let $\{a_1, \dots, a_n\}$ and $\{b_1, \dots, b_n\}$ the sets of finite values of f respectively of g.
$\tilde{f} = \sum_{j=1}^{n} a_j \chi_{\{x\in X;\ f(x)=a_j\}}$ and $\tilde{g} = \sum_{j=1}^{m} b_j \chi_{\{x\in X;\ g(x)=b_j\}}$, then
$\int_X f(x)d\,\mu(x) = \int_X \tilde{f}(x)d\,\mu(x)$ and
$\int_X g(x)d\,\mu(x) = \int_X \tilde{g}(x)d\,\mu(x)$ and $h = \tilde{g} - \tilde{f} \in \mathcal{E}^+$.

We deduce from 2) that

$$\int_X g(x)d\,\mu(x) = \int_X f(x)d\,\mu(x) + \int_X h(x)d\,\mu(x) \geq \int_X f(x)d\,\mu(x).$$

$\square$

Lemma

Let $(f_n)_n$ be an increasing sequence in $\mathscr{E}^+$, and if $g \in \mathscr{E}^+$ such that $g \leq \lim_{n\to+\infty} f_n$, then

$$\int_X g(x)d\,\mu(x) \leq \lim_{n\to+\infty} \int_X f_n(x)d\,\mu(x).$$

Proof

For $y \in g(X)$, let $E_y = \{x \in X;\ g(x) = y\}$. To prove the lemma it suffices to prove that for all $y \in g(X)$

$$\int_X g(x)\chi_{E_y}(x)d\,\mu(x) = y\mu(E_y) \leq \lim_{n\to+\infty} \int_X f_n(x)\chi_{E_y}(x)d\,\mu(x).$$

The result is trivial if $y = 0$. For $0 < t < y$, we set $A_n = E_y \cap \{x \in X;\ f_n(x) \geq t\}$. $(A_n)_n$ is an increasing sequence of measurable sets and $E_y = \lim_{n\to+\infty} A_n$, because for all $x \in E_y$, $f_n(x) > t$ for n large.

$$\begin{aligned} t\mu\{E_y \cap \{x \in X;\ f_n(x) > t\}\} &= \int_X t\chi_{E_y\cap\{x\in X;\ f_n(x)>t\}}(x)d\,\mu(x) \\ &\leq \int_X f_n(x)\chi_{E_y}(x)d\,\mu(x). \end{aligned}$$

So $t\mu(E_y) \leq \lim_{n\to+\infty} \int_X f_n(x)\chi_{E_y}(x)d\,\mu(x)$. This is for any $0 < t < y$, then

$$y\mu(E_y) \leq \lim_{n\to+\infty} \int_X f_n(x)\chi_{E_y}(x)d\,\mu(x).$$

To prove 4) of the proposition (23), we denote $g = \lim_{n\to+\infty} f_n$. Then $f_n \leq g$, $\forall n \in \mathbb{N}$ and the increasing sequence $\left(\int_X f_n(x) d\,\mu(x)\right)_n$ is bounded above by $\int_X g(x)\ d\,\mu(x)$.
For the other sense we applied the lemma (30).

□

Definition

Let f be a non-negative measurable function on a measure space $(X, \mathscr{A}, \mu)$, we define

$$\int_X f(x)d\,\mu(x) = \sup\{\int_X g(x)d\,\mu(x); g \leq f \text{ and } g \in \mathscr{E}^+\}$$

this is a non-negative number finite or infinite.

Remark

If f is a non-negative measurable function on a measure space $(X, \mathcal{B}, \mu)$, the theorem (16) yields the existence of an increasing sequence $(f_n)_n$ of $\mathscr{E}^+$ which converges to f. Then we have $\lim\limits_{n\to+\infty} \int_X f_n(x) d\,\mu(x) \leq \int_X f(x) d\,\mu(x)$. In the other hand for every function $g \in \mathscr{E}^+$ such that $g \leq f = \lim_{n\to+\infty} f_n$, we have from lemma (30) that $\int_X g(x) d\,\mu(x) \leq \lim\limits_{n\to+\infty} \int_X f_n(x) d\,\mu(x)$. So from the definition (34); $\int_X f(x) d\,\mu(x) \leq \lim\limits_{n\to+\infty} \int_X f_n(x) d\mu(x)$ and then $\int_X f(x) d\,\mu(x) = \lim\limits_{n\to+\infty} \int_X f_n(x) d\,\mu(x)$. This result is independent of the increasing sequence $(f_n)_n$ which converges to f. Then we have now the following theorem

Theorem

Let f and g be two non-negative measurable functions on a measure space $(X, \mathscr{A}, \mu)$, and let λ be a non-negative real number, then we have

1. $\displaystyle\int_X \lambda f(x)d\,\mu(x) = \lambda \int_X f(x)d\,\mu(x)$
2. $\displaystyle\int_X (f+g)(x)d\,\mu(x) = \int_X f(x)d\,\mu(x) + \int_X g(x)d\,\mu(x)$
3. If $f \leq g$ then $\displaystyle\int_X f(x)d\,\mu(x) \leq \int_X g(x)d\,\mu(x)$.

Proof

For the proof it is enough to consider two increasing sequences $(f_n)_n$ and $(g_n)_n$ of $\mathscr{E}^+$ which converge respectively to f and g, and then we apply the proposition (23).

□

Definition

Let f, g be two functions defined on $(X, \mathscr{A}, \mu)$. We say that $f = g$ almost everywhere, written $f = g$ a.e., if $\{x \in X;\ f(x) \neq g(x)\}$ is a null set. In particular if A is a measurable subset, then $\chi_A = 0$ a.e. if and only if $\mu(A) = 0$.

Definition

Let f be a function defined on $(X, \mathscr{A}, \mu)$. We say that f is defined almost everywhere on X if there exist a null subset N such that f is defined on the complementary of N.

Definition

A sequence $(f_n)_n$ of functions defined on $(X, \mathscr{A}, \mu)$ is said convergent almost everywhere to a function f if the set of x where the sequence $(f_n(x))_n$ is no convergent to $f(x)$is a null set.
We will denote by $\lim_{n\to+\infty} f_n$ any arbitrary measurable function f such that $(f_n)_n \longrightarrow f$ almost everywhere on X.

Proposition

Let f and g be two non-negative measurable functions defined on a measure space $(X, \mathscr{A}, \mu)$.

1. $\displaystyle\int_X f(x)d\,\mu(x) = 0$ if and only if $f = 0$ a.e.
2. If $f = g$ a.e then $\displaystyle\int_X f(x)d\,\mu(x) = \int_X g(x)\ d\,\mu(x)$.

Proof

1. We suppose that $\int_X f(x)d\,\mu(x) = 0$. If $A_n = \{x \in X;\ f(x) \geq \frac{1}{n}\}$, then $\chi_{A_n} \leq nf$ and $\int_X \chi_{A_n}(x)d\,\mu(x) = \mu(A_n) \leq n\int_X f(x)d\,\mu(x) = 0$. Then for all $n \in \mathbb{N}$; $\mu(A_n) = 0$. It results that $\{x;\ f(x) \neq 0\} = \bigcup_n A_n$ is a null set.

If $f = 0$ almost everywhere. The set $A = \{x \in X;\ f(x) \neq 0\}$ is a null. The function $g = \infty.\chi_A$ is a step function and $f \leq g$. Since $\int_X g(x)d\mu(x) = 0$, then $\int_X f(x)d\,\mu(x) = 0$.
(We can give an other solution based on the Monotone Convergence Theorem that will be proved:
We define $f_n = \inf(f, n)$ for all $n \in \mathbb{N}$. The sequence $(f_n)_n$ is increasing and $\int_X f_n(x)d\,\mu(x) = 0$, then it follows from the Monotone Convergence Theorem $\int_X f(x)d\,\mu(x) = 0$.)

2. We suppose that $f \leq g$. The function $h = g - f$ is defined a.e and equal to 0 a.e.
If $\int_X f(x)d\,\mu(x) = \int_X g(x)d\,\mu(x) = +\infty$, we have the desired result.
If $\int_X f(x)d\,\mu(x) < +\infty$, and $\int_X g(x)d\,\mu(x) < +\infty$, we have
$0 = \int_X h(x)d\,\mu(x) = \int_X g(x)d\,\mu(x) - \int_X f(x)d\,\mu(x)$.
Let now define the function $h = \inf(f, g)$. h is a non-negative measurable function and we have $h = f = g$ almost everywhere.

Since $h \leq f$ then $\int_X h(x)d\,\mu(x) = \int_X f(x)d\,\mu(x)$, and since $h \leq g$ then $\int_X h(x)d\,\mu(x) = \int_X g(x)d\,\mu(x)$. It results that $\int_X f(x)d\,\mu(x) = \int_X g(x)d\,\mu(x)$.

□

Definition

Let $f\colon X \longrightarrow \bar{\mathbb{R}}$ be a measurable function. If $f^+ = \sup(f, 0)$ and $f^- = \sup(-f, 0)$, then $f = f^+ - f^-$. The function f is called integrable with respect to the measure μ if and only if $\displaystyle\int_X f^+(x)d\,\mu(x)$ and $\displaystyle\int_X f^-(x)d\,\mu(x)$ are finite.

The integral of f will be denoted $\int_X f(x)\, d\,\mu(x) = \int_X f^+(x) d\,\mu(x) - \int_X f^-(x) d\,\mu(x)$, and if f is measurable and $\int_X f^+(x) d\,\mu(x) < +\infty$ or $\int_X f^-(x) d\,\mu(x) < +\infty$ we will denote of the same way $\int_X f(x) d\,\mu(x) = \int_X f^+(x) d\,\mu(x) - \int_X f^-(x) d\,\mu(x)$.
We define $\mathcal{L}^1(X)$ the space of integrable functions on X.

Proposition

The set $\mathcal{L}^1(X)$ is a vector space on $\mathbb{R}$ and the map $f \longmapsto \int_X f(x)d\,\mu(x)$ is a linear form on $\mathcal{L}^1(X)$ and we have $\Big|\int_X f(x)d\,\mu(x)\Big| \leq \int_X |f(x)|d\,\mu(x)$.

Proof

Let f and g be two integrable functions.

Since $\big|f+g\big| \leq |f|+|g|$, then $\displaystyle\int_X \big|f(x)+g(x)\big| d\,\mu(x)| \leq \int_X \big|f(x)\big| d\,\mu(x) + \int_X \big|g(x)\big|\ d\,\mu(x)$, and then $f+g \in\ L^1(X)$.

We have $f+g=(f+g)^+ - (f+g)^- = f^+ - f^- + g^+ - g^-$, then $(f+g)^+ + f^- + g^- = (f+g)^- + f^+ + g^+$. It follows that

$$\begin{aligned}\int_X (f+g)^+(x)d\,\mu(x) &+ \int_X f^-(x)d\,\mu(x) + \int_X g^-(x)\ d\,\mu(x)\\ &= \int_X (f+g)^-(x)d\,\mu(x) + \int_X f^+(x)\ d\,\mu(x)\\ &\quad + \int_X g^+(x)d\,\mu(x)\end{aligned}$$

and

$$\begin{aligned}
\int_X (f+g)(x) d\,\mu(x) &= \int_X (f+g)^+(x) d\,\mu(x) - \int_X (f+g)^-(x)\ d\,\mu(x) \\
&= \int_X f^+(x) d\,\mu(x) - \int_X f^-(x)\ d\,\mu(x) + \int_X g^+(x \\
&= \int_X f(x) d\,\mu(x) + \int_X g(x) d\,\mu(x).
\end{aligned}$$

The other properties are evident.

□

Corollary

1. If f is measurable and $a \leq f \leq b$ and $\mu(X) < +\infty$, then $f \in \mathcal{L}^1(X)$ and we have $a\mu(X) \leq \displaystyle\int_X f(x)\, d\,\mu(x) \leq b\mu(X)$.
2. If f is measurable and $g \in \mathcal{L}^1(X)$ and $f \leq g$, then $\displaystyle\int_X f(x) d\,\mu(x) \leq \int_X g(x) d\,\mu(x)$.
3. If E is a measurable null set, then $\displaystyle\int_E f(x) d\,\mu(x) = 0$ for any measurable function f.
4. Any bounded measurable function and equal to zero in the complementary of a subset of finite measure is integrable.

Remarks

1. Let f be an integrable function with respect to a measure μ. Then $\{x \in X;\ f(x) = \pm\infty\}$ is a null set.
2. On a measure space $(X, \mathscr{A}, \mu)$, the set of functions that are $f = 0$ a.e. is a vector space of $\mathcal{L}^1(X, \mathscr{A})$ closed under countable $(\sup, \inf)$. We denote $L^1(X, \mathscr{A})$ or $L^1(\mu)$ the quotient space $\mathcal{L}^1(X, \mathscr{A})$ by the space of null a.e functions. We call that $f = g$ in $L^1(X)$ if $f = g$ μ-almost everywhere.

Convergence Theorems

Monotone Convergence Theorem

Theorem

[Monotone Convergence Theorem or Beppo-Levi's Theorem]
Let $(f_n)_n$ be an increasing sequence of non-negative measurable functions on a measure space $(X, \mathcal{B}, \mu)$, then

$$\int_X \lim_{n\to+\infty} f_n(x) d\,\mu(x) = \lim_{n\to+\infty} \int_X f_n(x) d\,\mu(x).$$

Proof

For all integer n, there exists an increasing non-negative sequence $(\varphi_{n,j})_j$ of $\mathscr{E}^+$ which converges to f_n. For any j, set $\psi_j = \sup_{1\leq n\leq j} \varphi_{n,j}$.

Then the sequence $(\psi_j)_j \in \mathscr{E}^+$ is increasing because $\psi_j = \sup_{1\leq n\leq j} \varphi_{n,j} \leq \sup_{1\leq n\leq j} \varphi_{n,j+1} \leq \sup_{1\leq n\leq j+1} \varphi_{n,j+1} = \psi_{j+1}$.

We want to prove now that the sequence $(\psi_j)_j$ converges to f. We have for all $j \geq n$, $\varphi_{n,j} \leq \psi_j$, then $f_n = \lim_{j\to+\infty} \varphi_{n,j} \leq \lim_{j\to+\infty} \psi_j$, and then $f = \lim_{n\to+\infty} f_n \leq \lim_{j\to+\infty} \psi_j$. In the other hand, the inequalities $\varphi_{n,j} \leq f_n \leq f$ shows that $\psi_j \leq f$ and $\lim_{j\to+\infty} \psi_j \leq f$. The sequence $(\psi_j)_j$ is an increasing sequence of $\mathscr{E}^+$ and converges to f. Then $\int_X f(x)d\,\mu(x) = \lim_{j\to+\infty} \int_X \psi_j(x)d\,\mu(x)$. Moreover we have

$\psi_j \leq f_j$, then

$$\lim_{j\to+\infty}\int_X \psi_j(x)d\,\mu(x) \leq \lim_{j\to+\infty}\int_X f_j(x)\ d\,\mu(x) \leq \int_X f(x)d\,\mu(x),$$

which ends the proof of the theorem. □

Corollary

Let $(f_n)_n$ be a sequence of non-negative measurable functions on a measure space $(X, \mathscr{A}, \mu)$, then

$$\int_X \sum_{n=1}^{+\infty} f_n(x) d\,\mu(x) = \sum_{n=1}^{+\infty} \int_X f_n(x) d\,\mu(x)$$

Corollary

Let $(X, \mathscr{A}, \mu)$ be a measure space and let f be a non-negative measurable function. For all $A \in \mathscr{A}$, let $\tau(A) = \int_X f(x)\chi_A(x)d\,\mu(x)$. Then τ is a non-negative measure on $(X, \mathscr{A})$ called measure of density f with respect to the measure μ. The integral of a non-negative measurable function g by this measure is given by

$$\int_X g(x)\ d\,\tau(x) = \int_X f(x)g(x)d\,\mu(x).$$

Proof

Let $(A_n)_n$ be a finite or infinite sequence of measurable pairwise disjoints sets. We have $f\chi_{\cup_n A_n} = \sum_{n=1}^{+\infty} f\chi_{A_n}$. This which yields that

$$\begin{aligned}
\tau\,(\bigcup_n A_n) &= \int_X f(x)\chi_{\cup_n A_n}(x)d\,\mu(x) \\
&= \int_X \sum_{n=1}^{+\infty} f(x)\chi_{A_n}(x)d\,\mu(x) \\
&= \sum_{n=1}^{+\infty} \int_X f(x)\chi_{A_n}(x)d\,\mu(x).
\end{aligned}$$

The second part of the corollary is verified by any characteristic function χ_A of a measurable set A. Then it is valid for any simple non-negative function. By using the increasing continuity of the integrals, the result will be valid for non-negative measurable functions.

$\square$

Fatou's Lemma

Lemma

[Fatou's Lemma]
Let $(f_n)_n$ be a sequence of non-negative measurable functions on a measure space $(X, \mathscr{A}, \mu)$, then

$$\int_X \underline{\lim}_{n\to+\infty} f_n(x) d\,\mu(x) \leq \underline{\lim}_{n\to+\infty} \int_X f_n(x) d\,\mu(x).$$

Proof

$\underline{\lim}_{n\to+\infty} f_n = \lim_{n\to+\infty}(\inf_{j\geq n} f_j)$. We have $\displaystyle\int_X \inf_{j\geq n} f_j(x)\, d\,\mu(x) \leq \inf_{j\geq n}\int_X f_j(x) d\,\mu(x)$. The result follows from the Monotone Convergence Theorem.

□

Remark

Let $f_n = n^2\chi_{[0,\frac{1}{n}]}$, $\int_{\mathbb{R}} \underline{\lim}_{n\to+\infty} f_n(x)d\lambda(x) = 0$ and $\underline{\lim}_{n\to+\infty} \int_{\mathbb{R}} f_n(x)d\lambda(x) = +\infty$.

Dominate Convergence Theorem

Theorem

(Dominate Convergence Theorem (or Lebesgue Theorem)
Let $(f_n)_n$ be a sequence of measurable functions on a measure space $(X, \mathscr{A}, \mu)$. We assume that
i) the sequence $(f_n)_n$ converges almost everywhere on X to a measurable function f definite almost everywhere.
ii) There exist a non-negative integrable function g such that $|f_n| \leq g$ almost everywhere for all n. Then the sequence $(f_n)_n$ and the function f are integrable and we have

$$\int_X f(x)\ d\,\mu(x) \ = \ \lim_{n\to+\infty} \int_X f_n(x) d\,\mu(x).$$

The interest of the Dominated Convergence Theorem is that it does not require uniform convergence to permute the limit and the integral.

Theorem

Let $(f_n)_n$ be a sequence of measurable functions on a measure space $(X, \mathscr{A}, \mu)$. We assume that there exist a non-negative integrable function g such that for all n, $|f_n| \leq g$ almost everywhere. Then

$$\int_X \underline{\lim} f_n(x) d\,\mu(x) \leq \underline{\lim} \int_X f_n(x) d\,\mu(x) \tag{1}$$

$$\int_X \overline{\lim} f_n d\,\mu(x) \geq \overline{\lim} \int_X f_n(x) d\,\mu(x) \tag{2}$$

and if the sequence $(f_n)_n$ converges almost everywhere on X to a measurable function f defined almost everywhere, then $f \in L^1(X)$ and we have

$$\int_X f(x) d\,\mu(x) = \lim_{n\to+\infty} \int_X f_n(x) d\,\mu(x) \qquad (3)$$

Proof

The function g is finite almost everywhere on X because it is integrable. If we replace g by the function $g\chi_{\{x;\ g(x)<+\infty\}}$ this which not change the inequalities $|f_n| \leq g$ almost everywhere. Thus we can suppose that g is finite on X. We replace the sequence $(f_n)_n$ by the functions $f_n\chi_{\{|f_n|\leq g\}}$, this which not modified the integrals $\int_X f_n(x)d\,\mu(x)$ neither the equivalence classes $\lim_{n\to+\infty} f_n$ almost everywhere. Then we can suppose that $|f_n| \leq g$ on X. From these modifications, the functions $(f_n)_n$, $\overline{\lim} f_n$ and $\underline{\lim} f_n$ are finite and integrable on X. We apply the Fatou's lemma to the sequence $f_n + g$ we shall have

$$\int_X \underline{\lim}(f_n+g)(x)d\,\mu(x) \leq \underline{\lim} \int_X (f_n+g)(x)d\,\mu(x)$$

Since $\underline{\lim}_{n\to+\infty}(f_n+g) = (\underline{\lim}_{n\to+\infty} f_n\ ) \ +g$ on X, we shall have

$$\int_X \underline{\lim}_{n\to+\infty} f_n(x)d\,\mu(x) \le \underline{\lim}_{n\to+\infty}\int_X f_n(x)d\,\mu(x)$$

And by Fatou's lemma applied to the sequence $(-f_n+g)_n$ we shall have

$$\int_X \underline{\lim}_{n\to+\infty}(-f_n)(x)d\,\mu(x) \le \underline{\lim}_{n\to+\infty}\int_X -f_n(x)d\,\mu(x)$$

Then

$$\int_X \overline{\lim}_{n\to+\infty} f_n(x)d\,\mu(x) \geq \overline{\lim}_{n\to+\infty} \int_X f_n(x)d\,\mu(x)$$

The result follows easily. □

Exercise

Let f be an integrable function on $[0,+\infty[$. Find

$$\lim_{n\to+\infty}\int_0^{+\infty} e^{-n\sin^2 x} f(x)dx.$$

Solution Let $(f_n)_n$ be sequence defined by $f_n(x) = e^{-n\sin^2 x} f(x)$ on $[0,\infty[$. $A = \{x;\ f(x) = \pm\infty\} \cup \{n \in \mathbb{Z};\ n \geq 0\}$. For $x \notin A$, $\lim\limits_{n\to+\infty} f_n(x) = 0$ and $|f_n| \leq |f|$ which is integrable, then

$$\lim_{n\to+\infty}\int_0^{+\infty} e^{-n\sin^2 x} f(x)dx = 0.$$

Applications- Double Series

We consider the measure space $(\mathbb{N}, \mathscr{P}(\mathbb{N}), \mu)$ where μ is the measure defined by $\mu\{n\} = 1$ for all n of $\mathbb{N}$. In use the Dominate Convergence Theorem, we have the following result

Theorem

Let $(a_{m,n})_{m,n}$ be a double sequence of complex numbers such that
i) $\lim\limits_{n\to+\infty} a_{m,n} = a_m$ for all $m \in \mathbb{N}$,
ii) there exist a sequence $(b_m)_m$ of non-negative real numbers such that $\sum\limits_{m=1}^{+\infty} b_m < +\infty$ and $|a_{m,n}| \le b_m$ for all $n \in \mathbb{N}$.

Then we have $\lim\limits_{n\to+\infty} \sum\limits_{m=1}^{+\infty} a_{m,n} = \sum\limits_{m=1}^{+\infty} a_m.$

Integral Depending on Parameters

Let $(X, \mathscr{A}, \mu)$ be a measure space, and let E be a metric space.

Proposition

Let E be a metric space and $f \ \ E \times X \longrightarrow \mathbb{R}$ a function such that for all $t \in E$; the mapping $x \longmapsto f(t,x)$ is integrable. We define

$$F(t) = \int_X f(t,x) d\,\mu(x)$$

Let $a \in E$, we assume that
For almost all $x \in X$; the mapping $t \longmapsto f(t,x)$ is continuous at a.
There exist a neighborhood $V(a)$ of a and an integrable function g such that $\forall\ t \in V(a)$, $|f(t,.)| \leq g(.)$. Then F is continuous at a.

Proof

It suffices to apply the Dominate Convergence Theorem to the sequence $(f(a_n,.))_n$ for $n \in \mathbb{N}$; where $(a_n)_n$ is a sequence in $V(a)$ which converges to a. □

Exercise

Let f be an integrable function on $\mathbb{R}$ with respect to Lebesgue measure λ. We define

$$\widehat{f}(t) = \int_{\mathbb{R}} f(x)e^{-2i\pi xt}\, d\lambda(x)$$

Show that $\widehat{f}$ is continuous on $\mathbb{R}$.

Solution

Let g the function defined on $\mathbb{R}\times\mathbb{R}$ by $g(x,t)=f(x)e^{-2i\pi xt}$. The function $x\longmapsto g(x,t)$ is continuous a.e, the mapping $t\longmapsto g(x,t)$ is integrable and dominated by $|f|$ which is integrable. Then $\widehat{f}$ is continuous on $\mathbb{R}$.

Proposition

Let Ω be an open set of $\mathbb{R}$ and $f \ \Omega\times X \longrightarrow \mathbb{R}$ a function such that for all $t\in\Omega$; the mapping $x\longmapsto f(t,x)$ is integrable. We define

$$F(t)=\int_X f(t,x)d\,\mu(x).$$

We assume that
• for almost all $x\in X$; the mapping $t\longmapsto f(t,x)$ is derivable on Ω. We denote $\frac{\partial f}{\partial t}(t,x)$ its derivative,
• the function $f(t,.)$ is integrable on X and there exist a non-negative integrable function g such that for almost all $x\in X$, $|\frac{\partial f}{\partial t}(t,x)|\leq g(x)$ for all $t\in\Omega$. Then F is derivable on Ω and for all t in Ω

Proof

Let $a \in \Omega$ and $(h_n)_n$ be a sequence of real numbers converging to 0 and such that $a + h_n \in \Omega$. ($h_n \neq 0$, for all n). We define the sequence $(\varphi_n)_n$ by

$$\varphi_n(x) = \frac{f(a+h_n, x) - f(a,x)}{h_n}$$

For almost all $x \in X$, $\lim_{n\to\infty} \varphi_n(x) = \dfrac{\partial f}{\partial t}(a,x)$ and according to the mean value theorem, for such x we have $|\varphi_n(x)| \leq g(x)$. The Dominate Convergence Theorem yields that the function $x \longmapsto \dfrac{\partial f}{\partial t}(t,x)$ is integrable and

$$\int_X \frac{\partial f}{\partial t}(a,x)\, d\,\mu(x) = \lim_{n\to+\infty} \int_X \varphi_n(x) d\,\mu(x) = \lim_{n\to+\infty} \frac{F(a+h_n) - F(a)}{h_n}.$$

Examples

1. If for each $a \in \Omega$ there exists a neighborhood $V(a)$ and an integrable function g such that for almost all $x \in X$, $|\frac{\partial f}{\partial t}(t,x)| \leq g(x)$ for all $t \in V(a)$. Then F is differentiable on Ω and for all $t \in \Omega$

$$\frac{d}{dt}\int_X f(t,x)d\,\mu(x) = \int_X \frac{\partial f}{\partial t}(t,x)d\,\mu(x).$$

2. If in addition $\frac{\partial f}{\partial t}(t,x)$ is continuous, then F is $\mathcal{C}^1$.

Exercise

We consider the function F defined by

$$F(x) = \int_0^{+\infty} \frac{e^{-xt}}{1+t^2} dt$$

1. Show that F is continuous for $x \geq 0$ and $\lim_{x \to +\infty} F(x)$ and $\lim_{x \to 0} F(x)$ exist.
2. Show that F is of class $\mathcal{C}^2$ for $x > 0$ and verify the equation

$$y'' + y = \frac{1}{x} \tag{4}$$

Solution

1. The function $x \longmapsto \dfrac{e^{-xt}}{1+t^2}$ is continuous and $\displaystyle\lim_{x\to+\infty} \frac{e^{-xt}}{1+t^2} = 0$. Moreover this function is dominated by $\dfrac{1}{1+t^2}$ which is integrable. Then F is continuous for $x \geq 0$ and $\displaystyle\lim_{x\to+\infty} F(x) = 0$.

2. The function $x \longmapsto f(x,t) = \dfrac{e^{-xt}}{1+t^2}$ is $\mathcal{C}^1$,
$\dfrac{\partial f(x,t)}{\partial x} = \dfrac{-te^{-xt}}{1+t^2}$ is dominated by $\dfrac{te^{-at}}{1+t^2}$ for all $x \geq a > 0$ which is integrable. Then F is $\mathcal{C}^1$ on $[a,+\infty[$ for all $a > 0$, then F is $\mathcal{C}^1$ on $]0,+\infty[$.
$\dfrac{\partial^2 f(x,t)}{\partial x^2} = \dfrac{t^2e^{-xt}}{1+t^2}$. Moreover this function is dominated by $\dfrac{t^2e^{-at}}{1+t^2}$ for all $x \geq a > 0$ which is integrable. Then F is $\mathcal{C}^2$ on $[a,+\infty[$ for all $a > 0$, the F is $\mathcal{C}^2$ on $]0,+\infty[$ and $F''(x) =$
$\displaystyle\int_0^{+\infty} \frac{t^2e^{-xt}}{1+t^2}dt = -F(x) + \int_0^{+\infty} e^{-xt}dt = -F(x) + \frac{1}{x}.$

Exercise

Let f be an integrable function on $[0,1]$. Prove that

$$\lim_{n\to+\infty}\int_0^1 x^n f(x)dx = 0.$$

Solution

$|x^n\ f(x)| \leq |f(x)|$ which is integrable, and $\lim\limits_{n \to +\infty} x^n\ f(x) = 0$ a.e.
The result follows by the Dominate Convergence Theorem.

Exercise

Prove that

$$\lim_{n\to+\infty}\int_0^1 \frac{nx}{1+n^4x^4}dx = 0.$$

Solution

Let $(f_n)_n$ be the sequence defined on $[0,1]$ by $f_n(x) = \dfrac{nx}{1+n^4x^4}$. It is easy to prove that the sequence $(f_n)_n$ is uniformly bounded on $[0,1]$ by $\dfrac{3^{\frac{3}{4}}}{4}$ and $\lim\limits_{n\to+\infty} f_n(x) = 0$. Then by the Dominate Convergence Theorem

$$\lim_{n\to+\infty}\int_0^1 \frac{nx}{1+n^4x^4}dx = 0.$$

Exercise

Find $\lim_{n\to+\infty}\int_0^1 \frac{nx}{1+n^2x^4}dx$.

Solution

Let $(f_n)_n$ the sequence defined in $[0,1]$ by $f_n(x)=\frac{nx}{1+n^2x^4}$. $\lim_{n\to+\infty} f_n(x) = 0$ but $\int_0^1 \frac{nx}{1+n^2x^4}dx = \frac{1}{2}\int_0^n \frac{dt}{1+t^2}$, then $\lim_{n\to+\infty}\int_0^1 \frac{nx}{1+n^2x^4}dx = \frac{\pi}{4}$.

Comparison of Riemann and Lebesgue integrals

Riemann and Lebesgue Integrals

Let a and b two reals numbers, $a < b$. We consider the measure space $([a,b], \mathscr{B}^*, \lambda)$, where λ is the Lebesgue measure on $\mathbb{R}$ and $\mathscr{B}^*$ is the Lebesgue σ-algebra on $[a,b]$. For a bounded measurable function f on $[a,b]$, we denote $\int_a^b f(x)dx$ the Riemann integral for f on $[a,b]$ and $\int_{[a,b]} f(x)d\,\lambda(x)$ the Lebesgue integral, if they exist.

Let f be a bounded function on $[a,b]$. Then from the definition of the Riemann integral and the proprieties of the lower and upper Darboux sum of f, there exists an increasing sequence of partitions $(\sigma_n)_n$ of $[a,b]$ such that if $\sigma_n = \{x_0 = a, \dots, x_{p_n} = b\}$ the sequence $(\delta_n)_n$ defined by $\delta_n = \sup_{0\leq k\leq p_n-1} |x_{k+1} - x_k|$ converges to 0. (δ_n is called the norm of the partition). We denote

$$U(f) = \lim_{n\to+\infty} S(\sigma_n, f)$$

$$L(f) = \lim_{n\to+\infty} s(\sigma_n, f)$$

Let $(g_n)_n$ and $(h_n)_n$ be the sequences of simple functions defined by

$$g_n(x) = \begin{cases} m_k = \inf_{t\in[x_k,x_{k+1}[} f(t) \quad if \quad x_k \leq x < x_{k+1} \\ g_n(b) = f(b) \end{cases}$$

$$h_n(x) = \begin{cases} M_k = \sup_{t\in[x_k,x_{k+1}[} f(t) \quad if \quad x_k \leq x < x_{k+1} \\ h_n(b) = f(b) \end{cases}$$

The sequence $(g_n)_n$ is increasing and the sequence $(h_n)_n$ is decreasing. For $x \in [a, b]$, the sequence $(g_n)_n$ converges to a function g and the sequence $(h_n)_n$ converges to a function h. We remark that

$$U(\sigma_n, f) = \int_a^b h_n(x)dx = \int_{[a,b]} h_n(x)d\,\lambda(x).$$

$$L(\sigma_n, f) = \int_a^b g_n(x)dx = \int_{[a,b]} g_n(x)d\,\lambda(x).$$

Since g and h are measurable, it follows from the Monotone Convergence Theorem that

$$\lim_{n\to+\infty}\int_a^b g_n(x)dx = L(f) = \int_{[a,b]} g(x)d\,\lambda(x) \tag{5}$$

$$\lim_{n\to+\infty}\int_a^b h_n(x)dx = U(f) = \int_{[a,b]} f(x)d\,\lambda(x). \tag{6}$$

In the other hand $g(x) \leq f(x) \leq h(x)$ $\forall x \in [a, b]$.

Theorem

Let f be a bounded function on $[a, b]$.
a) If f is Riemann-integrable on $[a, b]$, then f is Lebesgue integrable on $[a, b]$ and

$$\int_{[a,b]} f(x)d\,\lambda(x) = \int_a^b f(x)dx.$$

b) f is Riemann-integral on $[a, b]$ if and only if, the set of discontinuity of f is a null set.
c) If the set of discontinuity of f is a null set, then f is Lebesgue integrable and

For the proof we need the following lemma

Lemma

Let f, g and h as above. For $x \in [a, b] \setminus \left(\bigcup_{n=1}^{+\infty} \sigma_n\right)$, $g(x) = h(x)$ if and only if f is continuous at x.

Proof of the lemma

Let $x \in [a, b] \setminus \left(\cup_{n=1}^{+\infty} \sigma_n\right)$ and $\delta_n = ||\sigma_n||$. The sequence $(\delta_n)_n$ converges to 0.
If f is continuous at x, then for $\varepsilon > 0, \exists \eta > 0$ such that $\forall t \in [a, b]$ and $|t - x| < \eta$, then $|f(x) - f(t)| < \varepsilon$.

Let n_0 such that $\forall n \geq n_0$, $\delta_{n_0} < \eta$.
For $n > n_0$, σ_n is a partition of $[a, b]$, then there exist $k \in \{0, \dots, p_n - 1\}$ such that $x_k < x < x_{k+1}$. Thus $\forall t \in]x_k, x_{k+1}[$, $|f(x) - f(t)| < \varepsilon$, then $h_n(x) = M_k \leq f(x) + \varepsilon$ and $g_n(x) = m_k \geq f(x) - \varepsilon$ and $h_n(x) - g_n(x) \leq \varepsilon$. This is for all $n \geq n_0$. Then $h(x) - g(x) \leq \varepsilon$ and this is for all $\varepsilon > 0$, which gives that $g(x) = h(x)$.

Conversely if $g(x) = h(x)$ and $x \notin (\bigcup_{n=1}^{\infty} \sigma_n)$. Since $g(x) \leq f(x) \leq h(x)$, then $f(x) = g(x) = h(x)$, $(g_n(x))_n$ and $(h_n(x))_n$ converges to $f(x)$.
Let $\varepsilon > 0$, it follows from the above result that there exists $n_0 \in \mathbb{N}$ such that $\forall n \geq n_0 \;\; 0 \leq f(x) - g_n(x) < \varepsilon$ and $0 \leq h_n(x) - f(x) < \varepsilon$. σ_{n_0} is a partition of $[a, b]$, then there exist $k \in \{0, \dots, p_{n_0} - 1\}$ such that $x \in [x_k, x_{k+1}[= I$. We have

$$h_{n_0}(x) - \varepsilon < f(x) < g_{n_0}(x) + \varepsilon$$

Moreover $h_{n_0}(x) = \sup_{t \in]x_k, x_{k+1}[} f(t)$ and $g_{n_0}(x) = \inf_{t \in]x_k, x_{k+1}[} f(t)$.
Then $\forall t \in I,\ f(t) - \varepsilon < f(x) < f(t) + \varepsilon$ this which yields that f is continuous at x. □

Proof of the Theorem

a) If f is Riemann-integrable on $[a, b]$, we have

$$L(f) = U(f) = \int_a^b f(x)dx$$

and from (5) and (6) we have $\int_{[a,b]} h(x)d\lambda(x) = \int_{[a,b]} g(x)d\lambda(x)$. Thus $\int_{[a,b]} (h(x) - g(x))d\lambda(x) = 0$. Moreover $h - g$ is a non-negative integrable function, then $h = g$ a.e. and $f = g$ a.e. Thus f is measurable and $\int_a^b f(x)dx = \int_{[a,b]} f(x)d\,\lambda(x)$.

b) The function f Riemann-integrable if and only if $U(f) = L(f)$. This is equivalent to $h = g$ a.e and the result is deduced from the previous lemma; indeed

The function f Riemann-integrable if and only if $h = g$ a.e which is equivalent to $\{x;\ h(x) \neq g(x)\} \cup (\bigcup_{n=1}^{\infty} \sigma_n)$ is a null set. This is equivalent to f continuous a.e on $[a, b]$.

c) If the set of discontinuity of f is a null set, then $\lim\limits_{n\to+\infty} g_n(x) = \lim\limits_{n\to+\infty} h_n(x) = f(x)$ at each point of continuity of f, then f is measurable and the Dominate Convergence Theorem yields

$$\lim_{n\to+\infty} \int_{[a,b]} g_n(x) d\lambda(x) = \int_{[a,b]} f(x) d\lambda(x)$$

$$\lim_{n\to+\infty} \int_{[a,b]} h_n(x) d\lambda(x) = \int_{[a,b]} f(x) d\lambda(x).$$

Thus f is Riemann integrable and

$$\int_a^b f(x) d\,\lambda(x) = \int_a^b f(x) dx.$$

We give now a new proof of the theorem (94)

Proposition

Let $f : [a, b] \to \mathbb{R}$ be a bounded function. f is Riemann integrable if and only if it is continuous almost everywhere on $[a, b]$.

Proof

a) Suppose that f is Riemann integrable. For $x \in [a, b]$, we define

$$g(x) = \sup_{\delta>0} \inf_{y\in[a,b],|y-x|\leq\delta} f(y) = \liminf_{y\to x} f(y),$$

$$h(x) = \inf_{\delta>0} \sup_{y\in[a,b],|y-x|\leq\delta} f(y) = \limsup_{y\to x} f(y).$$

f is continuous at x if and only if $g(x) = h(x)$. We have $g \leq f \leq h$. If σ is a partition of $[a, b]$, then $U(\sigma, g) \leq U(\sigma, f) \leq U(\sigma, h)$ and $L(\sigma, g) \leq s(\sigma, f) \leq s(\sigma, h)$. But $U(\sigma, f) = U(\sigma, h)$ and $L(\sigma, g) = s(\sigma, f)$, because on any open interval $]c, d[\subset [a, b]$ we have

$$\inf_{x\in]c,d[} g(x) = \inf_{x\in]c,d[} f(x), \quad \sup_{x\in]c,d[} f(x) = \sup_{x\in]c,d[} h(x).$$

It follows that

$$L(f) = L(g) \leq U(g) \leq U(f), \quad L(f) \leq L(h) \leq U(h) = U(f).$$

Since f is Riemann integrable, both g and h must be Riemann integrable, with integrals equal to $\int_a^b f(x)dx$. Then, they are both Lebesgue integrable, with the same integral. But $g \leq h$, so $g = h$ a.e. Now f is continuous at any point where g and h are equal, so f is continuous a.e.

b) Now suppose that f is continuous a.e. For $n \in \mathbb{N}$, let σ_n be the uniform partition of $[a, b]$ into 2^n intervals. Set

$$h_n(x) = \sup_{y \in]c,d[} f(y), \quad g_n(x) = \inf_{y \in]c,d[} f(y)$$

if $]c, d[$ is an open interval of σ_n containing x and $h_n(x) = g_n(x) = f(x)$ if $x \in \sigma_n$. Then $(g_n)_n$, $(h_n)_n$ are respectively increasing and decreasing sequences of functions and $L_(\sigma_n, f) = \int_a^b g_n(x)dx$, $U(\sigma_n, f) = \int_a^b h_n(x)dx$.
$\lim_{n\to\infty} g_n(x) = \lim_{n\to\infty} h_n(x) = f(x)$ at any point x at which f is continuous, so $f = \lim\limits_{n\to\infty} g_n = \lim\limits_{n\to\infty} h_n$ a.e. By Dominated Convergence Theorem, $\lim\limits_{n\to\infty} \displaystyle\int_a^b g_n(x)dx = \int_a^b f(x)dx = \lim\limits_{n\to\infty} \int h_n(x)dx$.
This means that $L(f) \geq \int_a^b f(x)dx \geq U(f)$ and f is Riemann integrable.

Case of Generalized Riemann Integral

Theorem

Let f be a locally Lebesgue-integrable function defined on an interval $]a, b[$. f is Lebesgue-integrable on $]a, b[$ if and only if the improper integral $\int_a^b f(x)dx$ is absolutely convergent and in this case the generalized Riemann integral and the Lebesgue integral coincide (i.e. $\int_a^b f(x)dx = \int_a^b f(x)d\,\lambda(x)$.)

Proof

We assume that $\int_a^b f(x)dx$ is absolutely convergent. We consider two sequences $(a_n)_n$ and $(b_n)_n$ of $]a, b[$ such that the sequence $(a_n)_n$ decreases to a and the sequence $(b_n)_n$ increases to b. Let $\varphi_n(x) = |f(x)|\chi_{[a_n,b_n]}$. The sequence $(\varphi_n)_n$ increases to $|f|\chi_{]a,b[}$. The functions φ_n are measurable then f is measurable. It follows from Monotone Convergence Theorem that

$$\lim_{n\to+\infty} \int_{\mathbb{R}} \varphi_n(x)d\,\lambda(x) = \int_a^b |f(x)|d\,\lambda(x).$$

Moreover from the previous Theorem $\int_{\mathbb{R}} \varphi_n(x)d\,\lambda(x) = \int_{a_n}^{b_n} |f(x)|dx$

and from the previous definition $\lim_{n\to+\infty} \int_{\mathbb{R}} \varphi_n(x)d\,\lambda(x) = \int_a^b |f(x)|dx.$

Conversely If f is Lebesgue-integrable on $]a, b[$, then $|f|$ is Lebesgue-integrable on $]a, b[$. Let $(a_n)_n$ and $(b_n)_n$ be two sequences in $]a, b[$ such that the sequence $(a_n)_n$ decreases to a and $(b_n)_n$ increases to b. By the Monotone Convergence Theorem

$$\lim_{n\to+\infty} \int_a^b \varphi_n(x) d\,\lambda(x) = \int_a^b |f(x)| d\,\lambda(x) < +\infty.$$

Moreover $\displaystyle\int_a^b \varphi_n(x) d\,\lambda(x) = \int_{a_n}^{b_n} |f(x)| dx$, then

$\displaystyle\lim_{n\to+\infty} \int_{a_n}^{b_n} |f(x)| dx$ exists in $\mathbb{R}$ and $\displaystyle\int_a^b |f(x)| dx < +\infty$. $\square$