4.4
Coordinates and Basis

Definition:

If $S = \{v_1, v_2, \ldots, v_n\}$ is a set of vectors in a finite-dimensional vector space V , then S is called a **basis** for V if:

(a) S spans V.

(b) S is linearly independent

EXAMPLE 1

Show that the vectors v_1 = (1, 2, 1), v_2 = (2, 9, 0), and v_3 = (3, 3, 4) forma basis forR^3.

EXAMPLE 2

Show that the matrices

$$M_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad M_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad M_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad M_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

form a basis for the vector space M_{22} of 2×2 matrices.

Coordinates Relative to a Basis

THEOREM: Uniqueness of Basis Representation

If $S = \{v_1, v_2, \ldots, v_n\}$ is a basis for a vector space V, then every vector **v** in V can be expressed in the form $\mathbf{v} = c_1 v_1 + c_2 v_2 + \cdots + c_n v_n$ in exactly one way.

Definition:

If S = $\{v_1, v_2, ..., v_n\}$ is a basis for a vector space V, and

$$v = c_1 v_1 + c_2 v_2 + \cdots + c_n v_n$$

is the expression for a vector v in terms of the basis S, then the scalars $c_1, c_2, \cdots, c_n$ are called the **coordinates** of v relative to the basis S. The vector $(c_1, c_2, \cdots, c_n)$ in $\boldsymbol{R}^n$ constructed from these coordinates is called the **coordinate** vector of v **relative** to S; it is denoted by

$$(v)_S = (c_1, c_2, \cdots, c_n)$$

EXAMPLE 3

(a)Find the coordinate vector for the polynomial

$\mathbf{p} = c_0 + c_1x + c_2x^2 + \ldots + c_nx^n$

relative to the standard basis for the vector space P_n

(b)Find the coordinate vector of

$$B=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

relative to the standard basis for

EXAMPLE 4

(a)We showed in Example 1 that the vectors

$$v_1 = (1, 2, 1),\ v_2 = (2, 9, 0),\ v_3 = (3, 3, 4)$$

form a basis for R^3. Find the coordinate vector of v = (5,−1, 9) relative to the basis S = $\{v_1, v_2, v_3\}$.

(b) Find the vector v in R^3 whose coordinate vector relative to S is $(v)_S = (-1, 3, 2)$.

Exercise Set 4.4

1. Use the method of Example 3 to show that the following set of vectors forms a basis for R^2.

$$\{(2, 1), (3, 0)\}$$

2. Use the method of Example 3 to show that the following set of vectors forms a basis for R^3.

$$\{(3, 1, -4), (2, 5, 6), (1, 4, 8)\}$$

3. Show that the following polynomials form a basis for P_2.

$$x^2 + 1, \quad x^2 - 1, \quad 2x - 1$$

4. Show that the following polynomials form a basis for P_3.

$$1 + x, \quad 1 - x, \quad 1 - x^2, \quad 1 - x^3$$

5. Show that the following matrices form a basis for M_{22}.

$$\begin{bmatrix} 3 & 6 \\ 3 & -6 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -8 \\ -12 & -4 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix}$$

6. Show that the following matrices form a basis for M_{22}.

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

7. In each part, show that the set of vectors is not a basis for R^3.
   (a) $\{(2, -3, 1), (4, 1, 1), (0, -7, 1)\}$
   (b) $\{(1, 6, 4), (2, 4, -1), (-1, 2, 5)\}$

8. Show that the following vectors do not form a basis for P_2.

$$1 - 3x + 2x^2, \quad 1 + x + 4x^2, \quad 1 - 7x$$

9. Show that the following matrices do not form a basis for M_{22}.

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 2 & -2 \\ 3 & 2 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}$$

10. Let V be the space spanned by $\mathbf{v}_1 = \cos^2 x$, $\mathbf{v}_2 = \sin^2 x$, $\mathbf{v}_3 = \cos 2x$.
    (a) Show that $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is not a basis for V.
    (b) Find a basis for V.

11. Find the coordinate vector of $\mathbf{w}$ relative to the basis $S = \{\mathbf{u}_1, \mathbf{u}_2\}$ for R^2.
    (a) $\mathbf{u}_1 = (2, -4)$, $\mathbf{u}_2 = (3, 8)$; $\mathbf{w} = (1, 1)$
    (b) $\mathbf{u}_1 = (1, 1)$, $\mathbf{u}_2 = (0, 2)$; $\mathbf{w} = (a, b)$

12. Find the coordinate vector of $\mathbf{w}$ relative to the basis $S = \{\mathbf{u}_1, \mathbf{u}_2\}$ for R^2.
    (a) $\mathbf{u}_1 = (1, -1)$, $\mathbf{u}_2 = (1, 1)$; $\mathbf{w} = (1, 0)$
    (b) $\mathbf{u}_1 = (1, -1)$, $\mathbf{u}_2 = (1, 1)$; $\mathbf{w} = (0, 1)$

13. Find the coordinate vector of $\mathbf{v}$ relative to the basis $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ for R^3.
    (a) $\mathbf{v} = (2, -1, 3)$; $\mathbf{v}_1 = (1, 0, 0)$, $\mathbf{v}_2 = (2, 2, 0)$, $\mathbf{v}_3 = (3, 3, 3)$
    (b) $\mathbf{v} = (5, -12, 3)$; $\mathbf{v}_1 = (1, 2, 3)$, $\mathbf{v}_2 = (-4, 5, 6)$, $\mathbf{v}_3 = (7, -8, 9)$

14. Find the coordinate vector of $\mathbf{p}$ relative to the basis $S = \{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\}$ for P_2.
    (a) $\mathbf{p} = 4 - 3x + x^2$; $\mathbf{p}_1 = 1$, $\mathbf{p}_2 = x$, $\mathbf{p}_3 = x^2$
    (b) $\mathbf{p} = 2 - x + x^2$; $\mathbf{p}_1 = 1 + x$, $\mathbf{p}_2 = 1 + x^2$, $\mathbf{p}_3 = x + x^2$