4.1
Real Vector Spaces

Vector Space Axioms

DEFINITION:

Let V be an arbitrary nonempty set of objects on which two operations are defined: addition, and multiplication by numbers called ***scalars***. By ***addition*** we mean a rule for associating with each pair of objects **u** and **v** in V an object **u + v**, called the ***sum*** of **u** and **v**; by ***scalar multiplication*** we mean a rule for associating with each scalar k and each object **u** in V an object k **u**, called the ***scalar multiple*** of **u** by k.

If the following axioms are satisfied by all objects **u, v, w** in V and all scalars k and m, then we call V a ***vector space*** and we call the objects in V ***vectors***.

1. If **u** and **v** are objects in V, then **u + v** is in V.

2. u + v = v + u

3. u + (v + w) = (u + v) + w

4. There is an object **0** in V, called a ***zero vector*** for V, such that **0 + u = u + 0 = u** for all **u** in V.

5. For each **u** in V, there is an object −**u** in V, called a ***negative*** of **u**, such that **u** + (−**u**) = (−**u**) + **u** = **0**.

6. If k is any scalar and **u** is any object in V, then k **u** is in V.

7. k(**u + v**) = k **u** + k **v**

8. ($k + m$)**u** = k **u** + m **u**

9. k(m **u**) = ($k\,m$)(**u**)

10. 1**u** = **u**

EXAMPLE 1

Let V consist of a single object, which we denote by **0**, and define

0 + **0** = **0** and k **0** = **0**

for all scalars k. It is easy to check that all the vector space axioms are satisfied. We call this the ***zero vector space***.

EXAMPLE 2

Rn Is a Vector Space

, and define the vector space operations on V to be the usual operations of Let $V= \boldsymbol{R}^{\boldsymbol{n}}$ addition and scalar multiplication of n-tuples; that is,

$u_n + v_n)$ $v_n)$ = $(u_1 + v_1$,$u_2 + v_2, \ldots,$ $u_n)$ + $(v_1$,$v_2, \ldots,$ $\mathbf{u} + \mathbf{v}$ = $(u_1$,$u_2, \ldots,$

k$u_n)$ $k\mathbf{u}$ = (ku_1,$ku_2, \ldots,$

is closed under addition and scalar multiplication The set $V= \boldsymbol{R}^{\boldsymbol{n}}$

EXAMPLE 3

The Vector Space of Infinite Sequences of Real Numbers

Let V consist of objects of the form **u** = $(u_1, u_2, \ldots, u_n, \ldots)$
in which $u_1, u_2, \ldots, u_n, \ldots$ is an infinite sequence of real numbers. We define two infinite sequences to be *equal* if their corresponding components are equal, and we define addition and scalar multiplication component twise by

u + **v** = $(u_1, u_2, \ldots, u_n, \ldots) + (v_1, v_2, \ldots, v_n \ldots) = (u_1 + v_1, u_2 + v_2, \ldots, u_n + v_n, \ldots)$

k **u** = $(ku_1, ku_2, \ldots, ku_n, \ldots)$

In the exercises we ask you to confirm that V with these operations is a vector space. We will denote this vector space by the symbol R^∞.

EXAMPLE 4

The Vector Space of 2 × 2 Matrices

Let V be the set of 2 × 2 matrices with real entries, and take the vector space operations on V to be the usual operations of matrix addition and scalar multiplication; that is,

$$\mathbf{u}+\mathbf{v}=\begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix}+\begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix}=\begin{bmatrix} u_{11}+v_{11} & u_{12}+v_{12} \\ u_{21}+v_{21} & u_{22}+v_{22} \end{bmatrix}$$

$$k\mathbf{u}=k\begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix}=\begin{bmatrix} ku_{11} & ku_{12} \\ ku_{21} & ku_{22} \end{bmatrix}$$

EXAMPLE 7

Let $V = R^2$ and define addition and scalar multiplication operations as follows: If
$\mathbf{u} = (u_1, u_2)$ and $\mathbf{v} = (v_1, v_2)$, then define
$\mathbf{u} + \mathbf{v} = (u_1 + v_1, u_2 + v_2)$
and if k is any real number, then define
$k\mathbf{u} = (k\, u_1, 0)$

For example, if $\mathbf{u} = (2, 4)$, $\mathbf{v} = (-3, 5)$, and $k = 7$, then
$\mathbf{u} + \mathbf{v} = (2 + (-3), 4 + 5) = (-1, 9)$
$k\,\mathbf{u} = 7\mathbf{u} = (7 \cdot 2, 0) = (14, 0)$
The addition operation is the standard one from R^2, but the scalar multiplication is not.

Some Properties of Vectors

THEOREM:

Let V be a vector space, **u** a vector in V, and k a scalar; then:

(a) $0\mathbf{u} = \mathbf{0}$

(b) $k\,\mathbf{0} = \mathbf{0}$

(c) $(-1)\mathbf{u} = -\mathbf{u}$

(d) If $k\,\mathbf{u} = \mathbf{0}$, then $k = 0$ or $\mathbf{u} = \mathbf{0}$.

Exercise Set 4.1

1. Let V be the set of all ordered pairs of real numbers, and consider the following addition and scalar multiplication operations on $\mathbf{u} = (u_1, u_2)$ and $\mathbf{v} = (v_1, v_2)$:

$$\mathbf{u} + \mathbf{v} = (u_1 + v_1, u_2 + v_2), \quad k\mathbf{u} = (0, ku_2)$$

   (a) Compute $\mathbf{u} + \mathbf{v}$ and $k\mathbf{u}$ for $\mathbf{u} = (-1, 2)$, $\mathbf{v} = (3, 4)$, and $k = 3$.

   (b) In words, explain why V is closed under addition and scalar multiplication.

   (c) Since addition on V is the standard addition operation on R^2, certain vector space axioms hold for V because they are known to hold for R^2. Which axioms are they?

   (d) Show that Axioms 7, 8, and 9 hold.

   (e) Show that Axiom 10 fails and hence that V is not a vector space under the given operations.

2. Let V be the set of all ordered pairs of real numbers, and consider the following addition and scalar multiplication operations on $\mathbf{u} = (u_1, u_2)$ and $\mathbf{v} = (v_1, v_2)$:

$$\mathbf{u} + \mathbf{v} = (u_1 + v_1 + 1, u_2 + v_2 + 1), \quad k\mathbf{u} = (ku_1, ku_2)$$

   (a) Compute $\mathbf{u} + \mathbf{v}$ and $k\mathbf{u}$ for $\mathbf{u} = (0, 4)$, $\mathbf{v} = (1, -3)$, and $k = 2$.

   (b) Show that $(0, 0) \neq \mathbf{0}$.

   (c) Show that $(-1, -1) = \mathbf{0}$.

   (d) Show that Axiom 5 holds by producing an ordered pair $-\mathbf{u}$ such that $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$ for $\mathbf{u} = (u_1, u_2)$.

   (e) Find two vector space axioms that fail to hold.

► In Exercises 3–12, determine whether each set equipped with the given operations is a vector space. For those that are not vector spaces identify the vector space axioms that fail. ◄

9. The set of all 2×2 matrices of the form

$$\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$$

with the standard matrix addition and scalar multiplication.

10. The set of all real-valued functions f defined everywhere on the real line and such that $f(1) = 0$ with the operations used in Example 6.

11. The set of all pairs of real numbers of the form $(1, x)$ with the operations

$$(1, y) + (1, y') = (1, y + y') \quad \text{and} \quad k(1, y) = (1, ky)$$

12. The set of polynomials of the form $a_0 + a_1 x$ with the operations

$$(a_0 + a_1 x) + (b_0 + b_1 x) = (a_0 + b_0) + (a_1 + b_1)x$$

and

$$k(a_0 + a_1 x) = (ka_0) + (ka_1)x$$

13. Verify Axioms 3, 7, 8, and 9 for the vector space given in Example 4.

14. Verify Axioms 1, 2, 3, 7, 8, 9, and 10 for the vector space given in Example 6.

15. With the addition and scalar multiplication operations defined in Example 7, show that $V = R^2$ satisfies Axioms 1–9.

16. Verify Axioms 1, 2, 3, 6, 8, 9, and 10 for the vector space given in Example 8.

17. Show that the set of all points in R^2 lying on a line is a vector space with respect to the standard operations of vector addition and scalar multiplication if and only if the line passes through the origin.