Math 106
Integral Calculus
Logarithm and Exponential Functions

Ibraheem Alolyan

King Saud University

Logarithm and Exponential Functions

1. Natural Logarithm Function
2. Exponential Function
3. Integration using natural logarithm and exponential functions

Natural Logarithm Function

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Definition

The natural logarithm function is defined by

$$\ln x = \int_1^x \frac{1}{t}\, dt$$

for every $x > 0$.

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for every $x > 0$.

- $\ln x < 0, \quad$ if $x < 1$
- $\ln 1 = 0$
- $\ln x > 0, \quad$ if $x > 1$

Natural Logarithm Function

Theorem

$$\frac{d}{dx}\ln x = \frac{1}{x}$$

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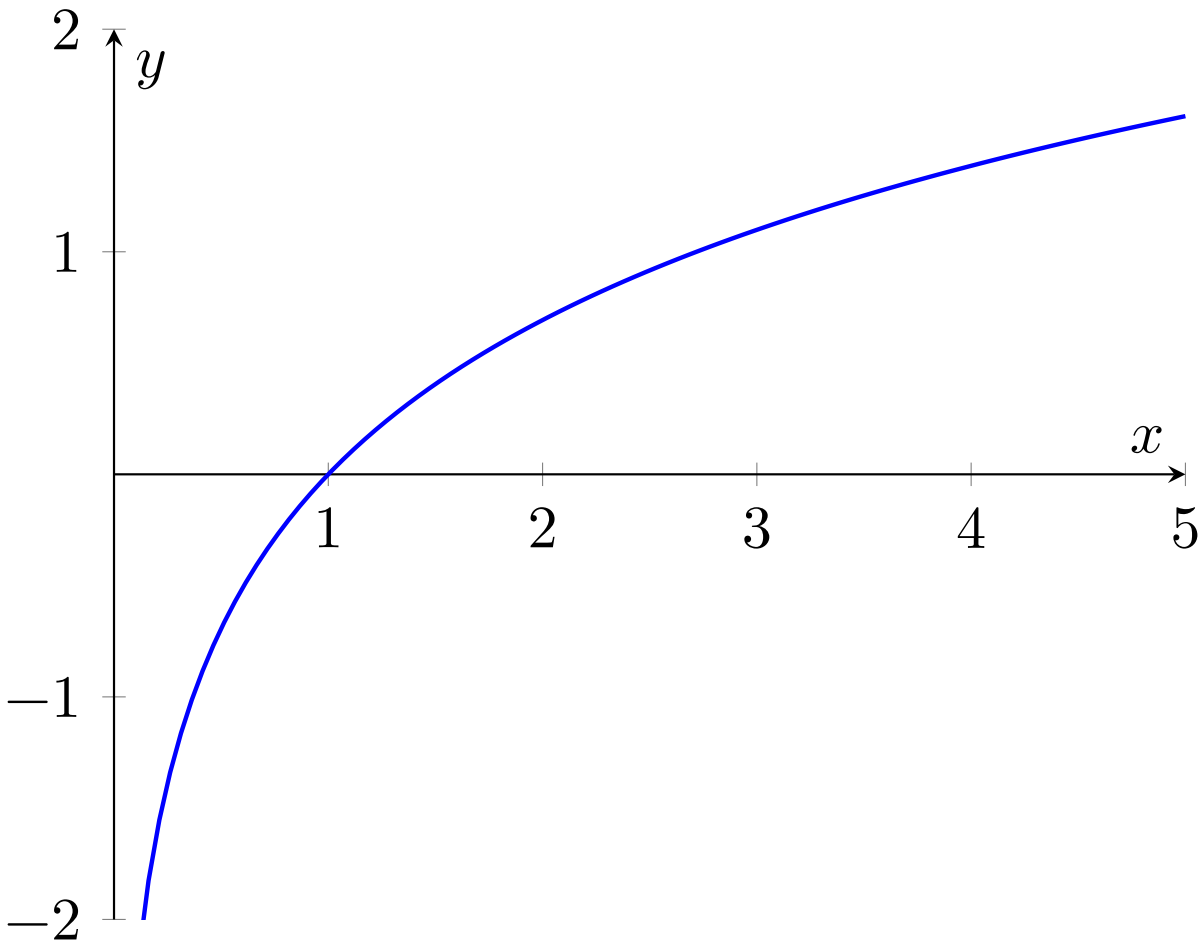
$y = \ln(x)$



Derivative of Natural Logarithm Function

Theorem

If g is differentiable, then

1. $$\frac{d}{dx}\ln g(x) = \frac{1}{g(x)}g'(x), \qquad g(x) > 0$$

2. $$\frac{d}{dx}\ln |g(x)| = \frac{1}{g(x)}g'(x), \qquad g(x) \neq 0$$

Laws of natural logarithm

Theorem

If $a, b > 0$, *and* $r \in \mathbb{Q}$ *then*

1. $\ln a\, b = \ln a + \ln b$
2. $\ln \dfrac{a}{b} = \ln a - \ln b$
3. $\ln a^r = r \ln a$

Derivative of Natural Logarithm Function

Example

Find the derivative of the following functions

1. $f(x) = \ln(x^2 + 1)$
2. $f(x) = \ln \sqrt{x + 1}$
3. $f(x) = \ln |x + \ln x|$
4. $f(x) = \sqrt{\ln x}$

Derivative of Natural Logarithm Function

Example

Find the derivative of the following functions

1. $f(x) = \ln[(x+1)(2x-4)]$
2. $f(x) = \ln \sqrt{\dfrac{x^2+1}{x^2-1}}$

Logarithmic differentiation of $y = f(x)$

Example

Find the derivative

$$y = \frac{(2x^2 + 1)^4}{\sqrt{x + 1}}$$

Exponential Function

Exponential Function

Definition

The natural exponential function exp is the inverse of the natural logarithm function. $y = exp\, x$ if and only if $x = \ln y$ $exp : \mathbb{R} \to (0, \infty)$

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Definition

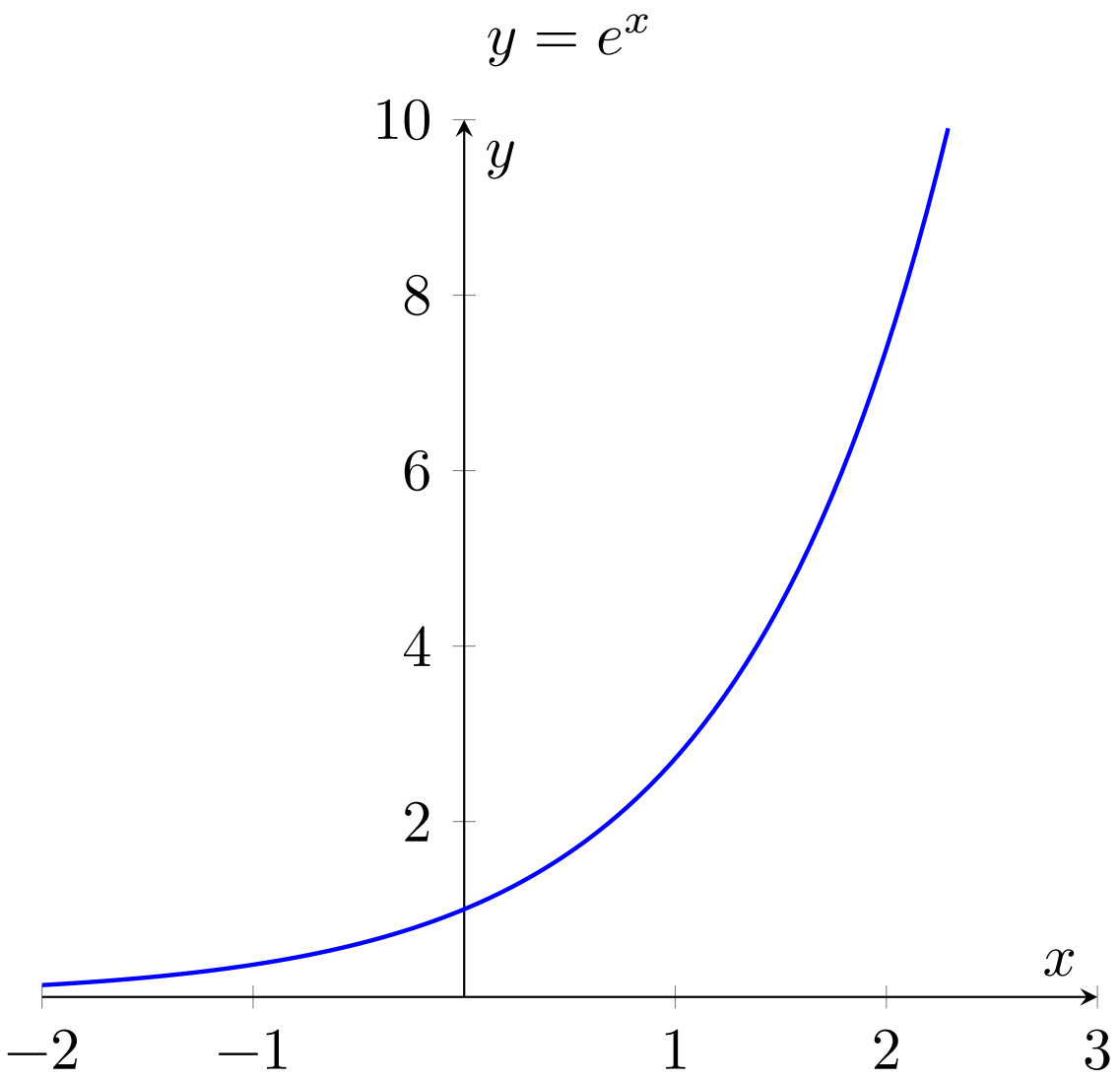
The letter e denotes the positive real number such that $\ln e = 1$ $(e \approx 2.718)$

Definition

If $x \in \mathbb{R}$, then

$$e^x = y \qquad \text{iff} \qquad \ln y = x$$

Exponential Function



Exponential Function

Theorem

1. $$\ln e^x = x, \qquad x \in \mathbb{R}$$

2. $$e^{\ln x} = x, \qquad x > 0$$

Exponential Function

Example

Simplify:

1. $\ln e^4$
2. $e^{\ln 3}$
3. $e^{3 \ln x}$
4. $e^{(x+2 \ln x)}$

Exponential Function

Theorem

If $a, b > 0$, *and* $r \in \mathbb{Q}$ *then*

1. $e^a e^b = e^{a+b}$
2. $\dfrac{e^a}{e^b} = e^{a-b}$
3. $\left(e^a\right)^r = e^{ar}$

Derivative of Exponential Function

Theorem

$$\frac{d}{dx}e^x = e^x$$

Derivative of Exponential Function

Example

If $f(x) = x^2 e^x$, find $f'(x)$.

Derivative of Exponential Function

Theorem

$$\frac{d}{dx}e^{g(x)} = e^{g(x)}g'(x)$$

Derivative of Exponential Function

Example

If $y = e^{\sqrt{x^2+x}}$, find $\dfrac{dy}{dx}$.

Derivative of Exponential Function

Example

If $y = x\, e^{\sin x}$, find $\dfrac{dy}{dx}$.

Derivative of Exponential Function

Example

If $y = \ln\left(x\, e^{x^2+x}\right)$, find $\dfrac{dy}{dx}$.

Integration using natural logarithm and exponential functions

Theorem

$$\int \frac{1}{x}\, dx = \ln|x| + c$$

Theorem

$$\int \frac{1}{x}\,dx = \ln|x| + c$$

Theorem

If $u = g(x) \neq 0$, and g is differentiable then

$$\int \frac{1}{u}\,du = \ln|u| + c$$

Example

Evaluate

1. $\int \frac{x}{2x^2+1}dx$
2. $\int_2^4 \frac{1}{9-2x}dx$

Example

Evaluate

1. $\int \frac{\sqrt{\ln x}}{x} dx$
2. $\int \frac{(6x + 9\sqrt{x})dx}{(x^2 + 2x\sqrt{x})}$

Example

Evaluate $\int \frac{1+\ln x}{\sqrt{x\ln x+2}}dx$

Exam problem

Example

Use the substitution $u = \ln|\sec x + \tan x|$ to compute the integral

$$\int (\sec x) \ln|\sec x + \tan x|\, dx$$

How to do it?

Example

$$\int \frac{1}{u+\sqrt{u}}\,dx$$

Theorem

$$\int e^x \, dx = e^x + c$$

Theorem

$$\int e^x \, dx = e^x + c$$

Theorem

If $u = g(x)$, and g is differentiable then

$$\int e^u \, du = e^u + c$$

Example

Evaluate

1. $\int e^{3x}dx$
2. $\int_1^2 \frac{e^{\frac{2}{x}}}{x^2}dx$
3. $\int \sqrt{e^{2x}}\,dx$

Exam Problem

Example

Evaluate the integral $\displaystyle\int \frac{(2+e^{4x})\,dx}{8x+e^{4x}}$

Integration of Trigonometric Frunctions

Theorem

1. $\int \tan x \, dx = \ln |\sec x| + c$
2. $\int \cot x \, dx = \ln |\sin x| + c$
3. $\int \sec x \, dx = \ln |\sec x + \tan x| + c$
4. $\int \csc x \, dx = \ln |\csc x - \cot x| + c$

Integration of Trigonometric Frunctions

Example

Evaluate

1. $\int x \cot x^2 \, dx$
2. $\int_0^{\frac{\pi}{2}} \tan \frac{x}{2} \, dx$

Integration of Trigonometric Frunctions

Example

Evaluate

1. $\int e^{3x} \sec e^{3x}\, dx$
2. $\int \frac{\cos^2 x}{\sin x}\, dx$

Integration of Trigonometric Frunctions

Example

Evaluate

1. $\displaystyle\int \frac{\cos x \sin x}{\sin^2 x - 1}\, dx$

Exam Problem

Example

Evaluate the integral $\int \frac{\tan\left(e^{-3x}\right)}{e^{3x}} dx$