UNIT 2

Determinants

Unit 2 · Sections 2.1 – 2.3

MATH 244 — Linear Algebra
Elementary Linear Algebra, Anton & Rorres, 11th edition
King Saud University · Department of Mathematics

LEARNING OUTCOMES

What you should be able to do

Section 2.1

- Find any minor and cofactor, reading its sign off the checkerboard.
- Evaluate a determinant by cofactor expansion along any row or column — the one with the most zeros.
- Write down a triangular determinant at sight; use the arrow technique for 2×2 and 3×3 only.

Section 2.2

- State exactly how each elementary row or column operation changes a determinant.
- Evaluate a determinant by reducing to triangular form, mixing row operations with cofactor expansion.

Section 2.3

- Use $\det(kA) = k^n \det(A)$, $\det(AB) = \det(A)\det(B)$, $\det\left(A^{-1}\right) = 1/\det(A)$ — and know there is no rule for $\det(A + B)$.
- Test invertibility by $\det(A) \neq 0$; build $\operatorname{adj}(A)$ and use $A^{-1} = \frac{1}{\det(A)} \operatorname{adj}(A)$.
- Solve a square system by Cramer's rule, and say when it applies.

SECTION 2.1

Determinants by Cofactor Expansion

GOAL — Attach to every square matrix a single number, defined one size at a time, and learn to compute it efficiently. Section 2.3 will show that this number decides invertibility.

1. Where the number $ad - bc$ comes from
2. Minors, cofactors and the checkerboard of signs
3. Cofactor expansion along any row or column
4. Choosing the row or column with the most zeros
5. Triangular matrices
6. The arrow technique for 2×2 and 3×3

The determinant of a 2×2 matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \qquad \det(A) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

- A is invertible exactly when $ad - bc \neq 0$, and then

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

> Two notations, one number: $\det(A)$ and $|A|$. The bars are NOT an absolute value — a determinant can be negative: $\begin{vmatrix} 3 & 1 \\ 4 & -2 \end{vmatrix} = (3)(-2) - (1)(4) = -10$.

WARNING — $\det(A)$ is a NUMBER, whereas A is a MATRIX.

Where the number $ad - bc$ comes from

- Solve the general 2×2 system by elimination:

$$a_{11}x_1 + a_{12}x_2 = b_1$$
$$a_{21}x_1 + a_{22}x_2 = b_2$$

- Multiply the equations by a_{22} and a_{12} and subtract to remove x_2; by a_{21} and a_{11} to remove x_1:

$$(a_{11}a_{22} - a_{12}a_{21})\,x_1 = b_1a_{22} - a_{12}b_2, \qquad (a_{11}a_{22} - a_{12}a_{21})\,x_2 = a_{11}b_2 - b_1a_{21}$$

> The SAME number $a_{11}a_{22} - a_{12}a_{21}$ multiplies both unknowns. When it is not zero, each unknown is forced — exactly one solution. This chapter builds that number for every square matrix: the DETERMINANT.

The right-hand sides are 2×2 determinants too. In Section 2.3 this observation becomes Cramer's rule.

DEFINITION 1

Minors and cofactors

- Determinants are defined one size at a time. For a 1×1 matrix, $\det[a_{11}] = a_{11}$; every larger size is then built from determinants one size smaller.

> If A is a square matrix, the MINOR of entry a_{ij}, written M_{ij}, is the determinant of the submatrix that remains after the i-th row and the j-th column of A are deleted. The number $C_{ij} = (-1)^{i+j} M_{ij}$ is the COFACTOR of entry a_{ij}.

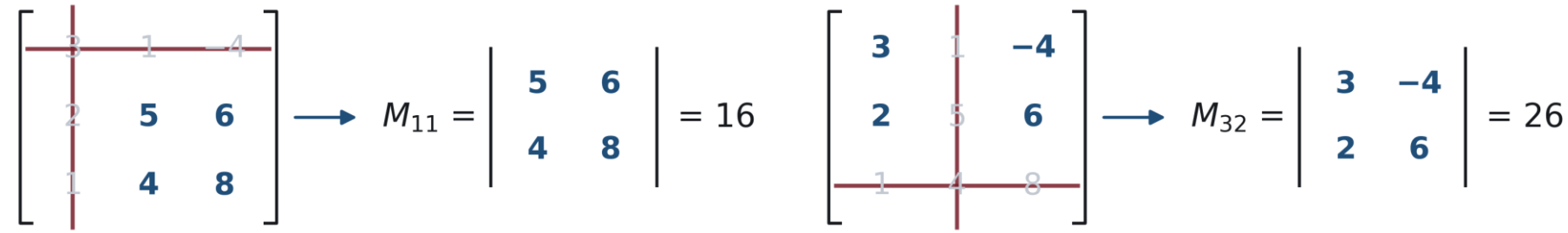
- Check on a 2×2: deleting row 1 and column 1 of $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ leaves $[d]$, so $M_{11} = C_{11} = d$; deleting row 1 and column 2 leaves $[c]$, so $M_{12} = c$ but $C_{12} = -c$.

Capital letters are used even though M_{ij} and C_{ij} are NUMBERS, not matrices.

EXAMPLE 1

Finding minors and cofactors

$$A = \begin{bmatrix} 3 & 1 & -4 \\ 2 & 5 & 6 \\ 1 & 4 & 8 \end{bmatrix}$$

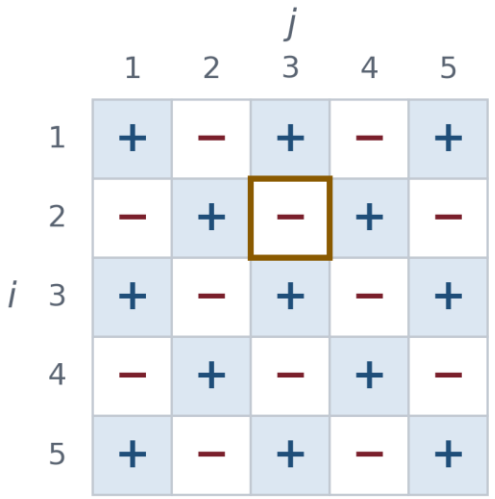


$$C_{11} = (-1)^{1+1} M_{11} = 16, \qquad C_{32} = (-1)^{3+2} M_{32} = -26$$

REMARK

The checkerboard of signs

- A minor and its cofactor are either EQUAL or OPPOSITE. Which one depends only on the position (i, j) of the entry:



position (2, 3)**:** $2 + 3 = 5$ is odd, so $C_{23} = -M_{23}$

sign of C_{ij} $= (-1)^{i+j}$

$+$ when $i + j$ is even, $-$ when $i + j$ is odd

So $C_{11} = M_{11}$, $C_{21} = -M_{21}$, $C_{22} = M_{22}$, … — compute the minor, then read the sign off the board. There is never any need to evaluate $(-1)^{i+j}$.

EXAMPLE 2

Four cofactor expansions of a 2×2 matrix

- For $A = [a_{ij}]$ of size 2×2 the checkerboard gives

$$C_{11} = a_{22}, \qquad C_{12} = -a_{21}, \qquad C_{21} = -a_{12}, \qquad C_{22} = a_{11}$$

- Multiply the entries of any ONE row or column by their own cofactors, and add:

	Cofactor expansion	Value
Along row 1	$a_{11}C_{11} + a_{12}C_{12}$	$a_{11}a_{22} - a_{12}a_{21}$
Along row 2	$a_{21}C_{21} + a_{22}C_{22}$	$-a_{21}a_{12} + a_{22}a_{11}$
Along column 1	$a_{11}C_{11} + a_{21}C_{21}$	$a_{11}a_{22} - a_{21}a_{12}$
Along column 2	$a_{12}C_{12} + a_{22}C_{22}$	$-a_{12}a_{21} + a_{22}a_{11}$

All four give $\det(A) = a_{11}a_{22} - a_{12}a_{21}$. In each one, the entries and their cofactors come from the SAME row or column.

Cofactor expansion along any row or column

> THEOREM 2.1.1 — If A is $n \times n$, then whichever row or column of A is chosen, multiplying its entries by the corresponding cofactors and adding always gives the SAME number.

- That number is the DETERMINANT of A, and the sums are its COFACTOR EXPANSIONS:

$$\det(A) = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj} \qquad (\text{along column } j)$$

$$\det(A) = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in} \qquad (\text{along row } i)$$

The text states the theorem without proof. The definition is INDUCTIVE: an $n \times n$ determinant is computed from $(n-1) \times (n-1)$ ones, and so on down to $\det[a_{11}] = a_{11}$.

EXAMPLES 3 AND 4

One determinant, two expansions

$$A = \begin{bmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{bmatrix}$$

Along the first row

$$\det(A) = 3\begin{vmatrix} -4 & 3 \\ 4 & -2 \end{vmatrix} - 1\begin{vmatrix} -2 & 3 \\ 5 & -2 \end{vmatrix} + 0\begin{vmatrix} -2 & -4 \\ 5 & 4 \end{vmatrix} = 3(-4) - (1)(-11) + 0 = -1$$

Along the first column

$$\det(A) = 3\begin{vmatrix} -4 & 3 \\ 4 & -2 \end{vmatrix} - (-2)\begin{vmatrix} 1 & 0 \\ 4 & -2 \end{vmatrix} + 5\begin{vmatrix} 1 & 0 \\ -4 & 3 \end{vmatrix} = 3(-4) - (-2)(-2) + 5(3) = -1$$

The first row needed only TWO cofactors, because its third entry is 0. Rule of thumb: expand along the row or column with the MOST ZEROS.

EXAMPLE 5

A smart choice of row or column

$$A = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 3 & 1 & 2 & 2 \\ 1 & 0 & -2 & 1 \\ 2 & 0 & 0 & 1 \end{bmatrix}$$

- Column 2 has three zeros: only the entry 1 in position $(2,2)$ survives, with sign $+$.

$$\det(A) = 1 \cdot \begin{vmatrix} 1 & 0 & -1 \\ 1 & -2 & 1 \\ 2 & 0 & 1 \end{vmatrix} = 1 \cdot (-2) \cdot \begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix} = -2(1+2) = -6$$

- In the 3×3 determinant, column 2 again has a single nonzero entry, -2 in position $(2,2)$, sign $+$. A 4×4 determinant cost ONE 2×2 determinant.

THEOREM 2.1.2

Triangular matrices: multiply the diagonal

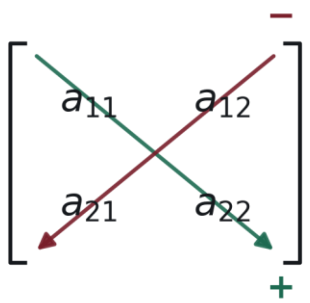
- Expand a lower triangular matrix along its FIRST ROW, again and again (Example 6): each time only the diagonal entry survives.

$$\begin{vmatrix} a_{11} & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ a_{31} & a_{32} & a_{33} & 0 \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & 0 & 0 \\ a_{32} & a_{33} & 0 \\ a_{42} & a_{43} & a_{44} \end{vmatrix} = a_{11}a_{22} \begin{vmatrix} a_{33} & 0 \\ a_{43} & a_{44} \end{vmatrix} = a_{11}a_{22}a_{33}a_{44}$$

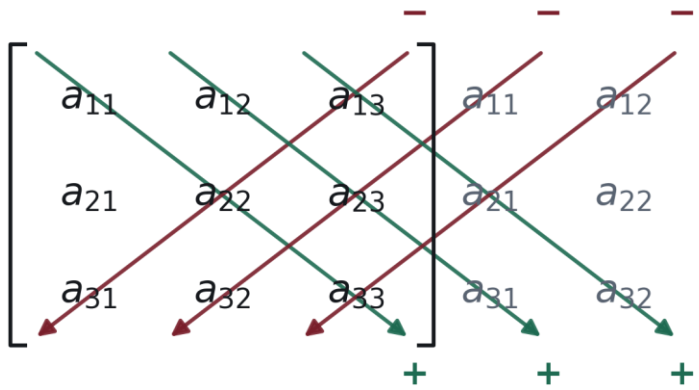
> If A is triangular — upper, lower, or diagonal — then $\det(A) = a_{11}a_{22}\cdots a_{nn}$, the product of the entries on the main diagonal.

$$\begin{vmatrix} -1 & 0 & 0 & 0 \\ 3 & 4 & 0 & 0 \\ 6 & 0 & 1/2 & 0 \\ 1 & 1 & 1 & 3 \end{vmatrix} = (-1)(4)(1/2)(3) = -6, \qquad \begin{vmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{vmatrix} = (2)(3)(4) = 24$$

The arrow technique — 2×2 and 3×3 only



2×2



3×3: recopy columns 1 and 2 on the right

rightward arrows: add

leftward arrows: subtract

- Multiply along each arrow; ADD the products on the rightward arrows and SUBTRACT those on the leftward arrows:

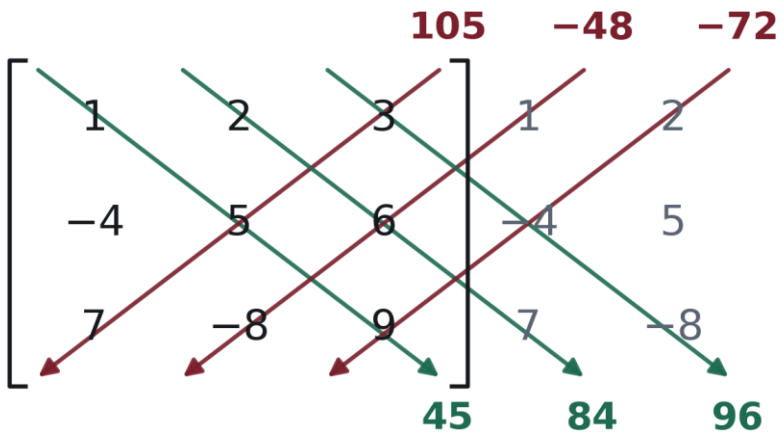
$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32}$$

WARNING — the arrow technique does NOT work for 4×4 or larger determinants.

EXAMPLE 7

The arrow technique in action

$$\begin{vmatrix} 3 & 1 \\ 4 & -2 \end{vmatrix} = (3)(-2) - (1)(4) = -10$$



add: $45 + 84 + 96 = 225$

subtract: $105 - 48 - 72 = -15$

$\det = 225 - (-15) = 240$

- Check it by a cofactor expansion along the first row: $1(93) - 2(-78) + 3(-3) = 93 + 156 - 9 = 240$.

CHECKPOINT 2.1

Suggested exercises

> Exercise Set 2.1: problems 1 – 31 (odd).

Before you move on, make sure you can

- Find any minor and cofactor, with the sign read off the checkerboard.
- Expand a determinant along any row or column — and choose the one with the most zeros.
- Write down the determinant of a triangular matrix at sight.
- Use the arrow technique on a 3×3 determinant, and never on a 4×4.
- Solve $\det(A) = 0$ for a parameter such as λ.

SECTION 2.2

Evaluating Determinants by Row Reduction

GOAL — Learn exactly how each elementary row operation changes a determinant, so that a row reduction to triangular form computes it — the method of choice for anything larger than 3×3.

1. Why a second method?
2. Zero rows, and $\det(A^T) = \det(A)$
3. How the three row operations change a determinant
4. Elementary matrices; proportional rows and columns
5. Evaluating a determinant by row reduction
6. Column operations, and mixing the two methods

Which of these would you rather evaluate?

$$|A| = \begin{vmatrix} 1 & -2 & 3 & 1 \\ 4 & -6 & 3 & 2 \\ -2 & 4 & -9 & -3 \\ 3 & -6 & 9 & 2 \end{vmatrix} \qquad |B| = \begin{vmatrix} 1 & -2 & 3 & 1 \\ 0 & 2 & -9 & -2 \\ 0 & 0 & -3 & -1 \\ 0 & 0 & 0 & -1 \end{vmatrix}$$

- $|B| = (1)(2)(-3)(-1) = 6$ at sight (Theorem 2.1.2). Cofactor expansion of $|A|$ needs four 3×3 determinants — and also gives 6.
- B came from A by adding multiples of row 1 to the other rows. This section explains why that cannot change the determinant.

	$n = 3$	$n = 5$	$n = 10$	$n = 25$
Cofactor expansion	9	205	6,235,300	$\approx 2.7 \times 10^{25}$
Row reduction	10	44	339	5,224

Multiplications for an $n \times n$ determinant (Larson, §3.2, extended to $n = 25$). At a trillion per second, the 25×25 cofactor expansion would take over 800,000 years.

Two quick facts

> THEOREM 2.2.1 — If A has a row of zeros or a column of zeros, then $\det(A) = 0$.

Proof: expand along that row or column; every term is $0 \cdot C_{ij}$.

> THEOREM 2.2.2 — If A is a square matrix, then $\det(A^T) = \det(A)$.

Proof: transposing turns rows into columns, so the expansion of A along any row is exactly the expansion of A^T along the matching column.

Consequence: every theorem about the ROWS of a determinant has a twin about its COLUMNS.

THEOREM 2.2.3

How row operations change a determinant

- Let A be $n \times n$, and let B come from A by ONE elementary row operation:

	Operation on A	Course symbol	Effect on the determinant
Scale	multiply row i by k	$M_i^{(k)}$	$\det(B) = k\det(A)$
Interchange	swap rows i and j	$I_{i,j}$	$\det(B) = -\det(A)$
Replace	add k times row i to row j	$A_{i,j}^{(k)}$	$\det(B) = \det(A)$

- By Theorem 2.2.2, the same three rules hold for COLUMN operations.

> Only SCALING brings in a factor; only an INTERCHANGE brings in a sign; the REPLACEMENT — the workhorse of elimination — changes nothing.

See the rules on 2×2, then prove one

Each rule on a 2×2 determinant

- Interchange $I_{1,2}$: $\begin{vmatrix} 2 & -3 \\ 1 & 4 \end{vmatrix} = 11$ becomes $\begin{vmatrix} 1 & 4 \\ 2 & -3 \end{vmatrix} = -11.$
- Replacement $A_{1,2}^{(-2)}$: $\begin{vmatrix} 1 & -3 \\ 2 & -4 \end{vmatrix} = 2$ becomes $\begin{vmatrix} 1 & -3 \\ 0 & 2 \end{vmatrix} = 2.$
- Scaling $M_1^{(1/2)}$: $\begin{vmatrix} 2 & -8 \\ -2 & 9 \end{vmatrix} = 2$ becomes $\begin{vmatrix} 1 & -4 \\ -2 & 9 \end{vmatrix} = 1.$

Proof of the scaling rule, for 3×3

$$\begin{vmatrix} ka_{11} & ka_{12} & ka_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = ka_{11}C_{11} + ka_{12}C_{12} + ka_{13}C_{13} = k\det(A)$$

Expand along row 1. The cofactors C_{11}, C_{12}, C_{13} never look at row 1, so they are the same for both matrices.

Determinants of elementary matrices

- Apply Theorem 2.2.3 to $A = I_n$, whose determinant is 1. If E is an $n \times n$ elementary matrix, then

> $\det(E) = k$ if E comes from $M_i^{(k)}$; $\det(E) = -1$ if E comes from $I_{i,j}$; $\det(E) = 1$ if E comes from $A_{i,j}^{(k)}$.

$$\begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} = 3, \qquad \begin{vmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{vmatrix} = -1, \qquad \begin{vmatrix} 1 & 0 & 0 & 7 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} = 1$$

- Left to right: $M_2^{(3)}$, $I_{1,4}$ and $A_{4,1}^{(7)}$ applied to I_4.

An elementary matrix NEVER has determinant zero — consistent with Theorem 1.5.2: every elementary matrix is invertible.

Proportional rows or columns

> If A has two proportional rows or two proportional columns, then $\det(A) = 0$.

Proof: add a suitable multiple of one of the two rows to the other. That creates a row of zeros and — being a replacement — does not change the determinant. Now apply Theorem 2.2.1.

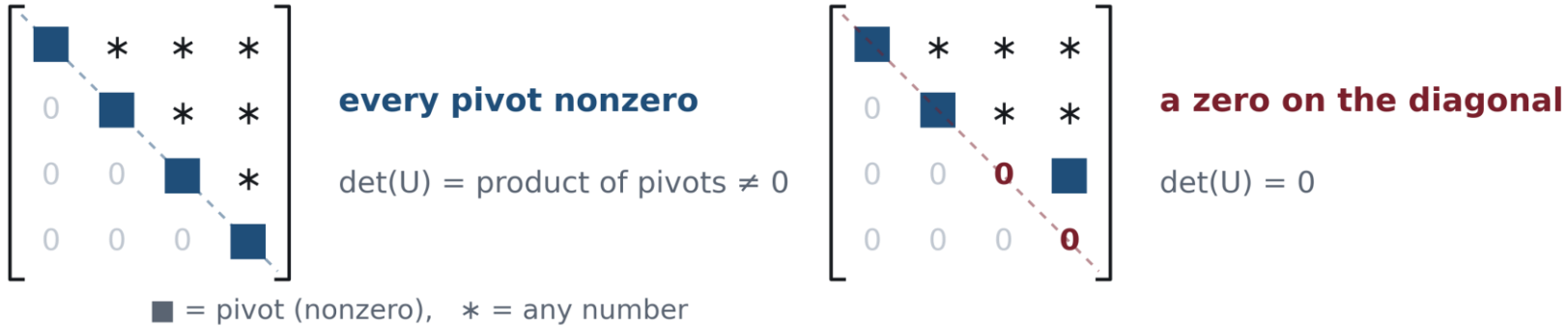
$$\begin{bmatrix} -1 & 4 \\ -2 & 8 \end{bmatrix}, \qquad \begin{bmatrix} 1 & -2 & 7 \\ -4 & 8 & 5 \\ 2 & -4 & 3 \end{bmatrix}, \qquad \begin{bmatrix} 3 & -1 & 4 & -5 \\ 6 & -2 & 5 & 2 \\ 5 & 8 & 1 & 4 \\ -9 & 3 & -12 & 15 \end{bmatrix}$$

- Row 2 is twice row 1; column 2 is -2 times column 1; row 4 is -3 times row 1. All three determinants are 0 — nothing to compute.

METHOD

Evaluating a determinant by row reduction

- Reduce A to an echelon form U — which is upper triangular — keeping track of what each step does to the determinant. Then multiply the diagonal of U.



$$\det(A) = \begin{cases} (-1)^r \times (\text{product of the pivots of } U) & \text{if } A \text{ is invertible} \\ 0 & \text{if } A \text{ is not invertible} \end{cases}$$

Here U is reached by replacements and r interchanges (Lay, §3.2).

> Write the operation above the equals sign, and change the factor in front to compensate: an interchange costs -1, taking k out of a row costs k, a replacement costs nothing.

EXAMPLE

Replacements only: nothing to keep track of

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 10 \end{bmatrix}$$

$$|A| \overset{A_{1,2}^{(-4)},\, A_{1,3}^{(-7)}}{=} \begin{vmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & -6 & -11 \end{vmatrix} \overset{A_{2,3}^{(-2)}}{=} \begin{vmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & 0 & 1 \end{vmatrix} = (1)(-3)(1) = -3$$

- Only replacements were used, so the determinant never changed: it is simply the product of the diagonal of the echelon form.

Example 3, next, needs all three kinds of operation.

EXAMPLE 3

All three operations at once

$$A = \begin{bmatrix} 0 & 1 & 5 \\ 3 & -6 & 9 \\ 2 & 6 & 1 \end{bmatrix}$$

$$|A| \overset{I_{1,2}}{=} -\begin{vmatrix} 3 & -6 & 9 \\ 0 & 1 & 5 \\ 2 & 6 & 1 \end{vmatrix} \overset{M_1^{(1/3)}}{=} -3\begin{vmatrix} 1 & -2 & 3 \\ 0 & 1 & 5 \\ 2 & 6 & 1 \end{vmatrix}$$

$$\overset{A_{1,3}^{(-2)}}{=} -3\begin{vmatrix} 1 & -2 & 3 \\ 0 & 1 & 5 \\ 0 & 10 & -5 \end{vmatrix} \overset{A_{2,3}^{(-10)}}{=} -3\begin{vmatrix} 1 & -2 & 3 \\ 0 & 1 & 5 \\ 0 & 0 & -55 \end{vmatrix} = (-3)(1)(1)(-55) = 165$$

- The interchange put a $-$ in front; taking 3 out of row 1 put a 3 in front; the two replacements changed nothing.

EXAMPLE 4

A column operation can finish the job in one step

$$A = \begin{bmatrix} 1 & 0 & 0 & 3 \\ 2 & 7 & 0 & 6 \\ 0 & 6 & 3 & 0 \\ 7 & 3 & 1 & -5 \end{bmatrix}$$

- Add -3 times column 1 to column 4. That is a column replacement, so the determinant does not change — and the matrix becomes lower triangular:

$$\det(A) = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 2 & 7 & 0 & 0 \\ 0 & 6 & 3 & 0 \\ 7 & 3 & 1 & -26 \end{vmatrix} = (1)(7)(3)(-26) = -546$$

Keep an eye open for column operations: row reduction would have taken several steps here.

EXAMPLE 5

Row operations and cofactor expansion together

$$A = \begin{bmatrix} 3 & 5 & -2 & 6 \\ 1 & 2 & -1 & 1 \\ 2 & 4 & 1 & 5 \\ 3 & 7 & 5 & 3 \end{bmatrix}$$

$$|A| \overset{A_{2,1}^{(-3)},\, A_{2,3}^{(-2)},\, A_{2,4}^{(-3)}}{=} \begin{vmatrix} 0 & -1 & 1 & 3 \\ 1 & 2 & -1 & 1 \\ 0 & 0 & 3 & 3 \\ 0 & 1 & 8 & 0 \end{vmatrix} = - \begin{vmatrix} -1 & 1 & 3 \\ 0 & 3 & 3 \\ 1 & 8 & 0 \end{vmatrix}$$

$$\overset{A_{1,3}^{(1)}}{=} - \begin{vmatrix} -1 & 1 & 3 \\ 0 & 3 & 3 \\ 0 & 9 & 3 \end{vmatrix} = -(-1) \begin{vmatrix} 3 & 3 \\ 9 & 3 \end{vmatrix} = 9 - 27 = -18$$

- Row operations create zeros; a cofactor expansion along the cleared column then cuts the size by one. The first $-$ is the checkerboard sign of position $(2,1)$.

EXAMPLE

Letters instead of numbers

$$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} \overset{A_{1,2}^{(-a)},\ A_{1,3}^{(-a^2)}}{=} \begin{vmatrix} 1 & 1 & 1 \\ 0 & b-a & c-a \\ 0 & b^2-a^2 & c^2-a^2 \end{vmatrix} = \begin{vmatrix} b-a & c-a \\ b^2-a^2 & c^2-a^2 \end{vmatrix}$$

$$= (b-a)(c-a)\begin{vmatrix} 1 & 1 \\ b+a & c+a \end{vmatrix} = (b-a)(c-a)(c-b)$$

- Expand along column 1; then take $b-a$ out of column 1 and $c-a$ out of column 2, using $b^2-a^2=(b-a)(b+a)$.

> Read the answer back: the determinant is 0 exactly when two of a, b, c are equal — exactly when the matrix has two equal columns. (Exercise Set 2.2, Exercise 23.)

SECTION 2.3

Properties of Determinants; Cramer's Rule

GOAL — See how the determinant treats scalar multiples, sums, products and inverses; turn $\det(A) \neq 0$ into the invertibility test; and write A^{-1} and the solution of $Ax = b$ as explicit formulas.

1. $\det(kA) = k^n \det(A)$ — and why sums have no rule
2. The determinant test for invertibility
3. $\det(AB) = \det(A)\det(B)$ and $\det\left(A^{-1}\right) = 1/\det(A)$
4. The adjoint, and the formula for A^{-1}
5. Cramer's rule
6. The Equivalence Theorem — seven statements

FORMULA

The determinant of a scalar multiple

> If A is $n \times n$ and k is a scalar, then $\det(kA) = k^n \det(A)$.

Each of the n rows of kA carries a common factor k; take it out, one row at a time:

$$\begin{vmatrix} ka_{11} & ka_{12} & ka_{13} \\ ka_{21} & ka_{22} & ka_{23} \\ ka_{31} & ka_{32} & ka_{33} \end{vmatrix} = k^3 \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

Example

$$A = \begin{bmatrix} 10 & -20 & 40 \\ 30 & 0 & 50 \\ -20 & -30 & 10 \end{bmatrix}, \qquad \begin{vmatrix} 1 & -2 & 4 \\ 3 & 0 & 5 \\ -2 & -3 & 1 \end{vmatrix} = 5 \qquad \Rightarrow \qquad |A| = 10^3 \cdot 5 = 5000$$

EXAMPLE 1

$$\det(A+B) \neq \det(A) + \det(B)$$

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}, \qquad B = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}, \qquad A+B = \begin{bmatrix} 4 & 3 \\ 3 & 8 \end{bmatrix}$$

$$\det(A) = 1, \qquad \det(B) = 8, \qquad \det(A+B) = 23 \neq 1+8$$

> There is NO simple rule for the determinant of a sum. The determinant is additive in ONE row at a time, with the other rows held fixed — that is the next theorem.

THEOREM 2.3.1

Adding determinants that differ in one row

> Let A, B, C be $n \times n$ matrices that differ only in row r, and suppose row r of C is the sum of rows r of A and B. Then $\det(C) = \det(A) + \det(B)$. The same holds for columns.

Example — only row 2 differs

$$\begin{vmatrix} 1 & -2 & 2 \\ 0 & 3 & 2 \\ 1 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 1 & -2 & 2 \\ -1 & 1 & 2 \\ 1 & 0 & 1 \end{vmatrix} + \begin{vmatrix} 1 & -2 & 2 \\ 1 & 2 & 0 \\ 1 & 0 & 1 \end{vmatrix} \qquad -7 = -7 + 0$$

Why: expand all three along row 2. The cofactors C_{21}, C_{22}, C_{23} never look at row 2, so they are the SAME for all three matrices — and $(0,3,2) = (-1,1,2) + (1,2,0)$ does the rest.

LEMMA 2.3.2

Multiplying by an elementary matrix

> If B is $n \times n$ and E is an $n \times n$ elementary matrix, then $\det(EB) = \det(E)\det(B)$.

Proof — the scaling case

If E comes from $M_i^{(k)}$, then EB is B with row i multiplied by k (Theorem 1.5.1). So $\det(EB) = k\det(B)$ by Theorem 2.2.3, and $k = \det(E)$ by Theorem 2.2.4. The interchange and replacement cases follow the same pattern.

Used repeatedly, the lemma gives

$$\det(E_1 E_2 \cdots E_r B) = \det(E_1)\det(E_2)\cdots\det(E_r)\det(B)$$

THEOREM 2.3.3

The determinant test for invertibility

> A square matrix A is invertible if and only if $\det(A) \neq 0$.

Proof

Let $R = E_r \cdots E_2 E_1 A$ be the reduced row echelon form of A. By the lemma, $\det(R) = \det(E_r) \cdots \det(E_1)\det(A)$, and no $\det(E_i)$ is 0 — so $\det(A)$ and $\det(R)$ are zero together or nonzero together.

If A is invertible, $R = I$ and $\det(R) = 1$. If not, R has a row of zeros and $\det(R) = 0$. ■

Example 3

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & 1 \\ 2 & 4 & 6 \end{bmatrix} \qquad \text{rows 1 and 3 are proportional} \;\Rightarrow\; \det(A) = 0 \;\Rightarrow\; A \text{ is not invertible}$$

EXAMPLE

Which system has exactly one solution?

(a)
$$\begin{aligned} 2x_2 - x_3 &= -1 \\ 3x_1 - 2x_2 + x_3 &= 4 \\ 3x_1 + 2x_2 - x_3 &= -4 \end{aligned}$$

$$\begin{vmatrix} 0 & 2 & -1 \\ 3 & -2 & 1 \\ 3 & 2 & -1 \end{vmatrix} = 0$$

(b)
$$\begin{aligned} 2x_2 - x_3 &= -1 \\ 3x_1 - 2x_2 + x_3 &= 4 \\ 3x_1 + 2x_2 + x_3 &= -4 \end{aligned}$$

$$\begin{vmatrix} 0 & 2 & -1 \\ 3 & -2 & 1 \\ 3 & 2 & 1 \end{vmatrix} = -12$$

- The two systems differ in ONE coefficient. In (b) the coefficient matrix is invertible, so there is exactly one solution (Theorem 1.6.4).
- In (a) it is not, so (a) has either NO solution or INFINITELY many — the determinant alone cannot say which. (Here: none.)

THEOREM 2.3.4

The determinant of a product

> If A and B are square matrices of the same size, then $\det(AB) = \det(A)\det(B)$.

Proof

If A is not invertible, neither is AB (Theorem 1.6.5), so both sides are 0.

If A is invertible, then $A = E_1 E_2 \cdots E_r$ (Theorem 1.5.3), and the lemma, used twice, gives $\det(AB)$ $= \det(E_1)\cdots\det(E_r)\det(B) = \det(E_1 \cdots E_r)\det(B) = \det(A)\det(B)$. ■

Example 4

$$A = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} -1 & 3 \\ 5 & 8 \end{bmatrix}, \quad AB = \begin{bmatrix} 2 & 17 \\ 3 & 14 \end{bmatrix}: \qquad \det(A)\det(B) = (1)(-23) = -23 = \det(AB)$$

Compare Example 1: the determinant respects PRODUCTS, but not SUMS.

THEOREM 2.3.5

The determinant of an inverse

> If A is invertible, then $\det\left(A^{-1}\right) = \frac{1}{\det(A)}$.

Proof: $A^{-1}A = I$, so $\det\left(A^{-1}\right)\det(A) = \det(I) = 1$; divide by $\det(A) \neq 0$. ■

The toolkit so far — A and B are $n \times n$

	Rule
Transpose	$\det\left(A^T\right) = \det(A)$
Scalar multiple	$\det(kA) = k^n \det(A)$
Product	$\det(AB) = \det(A)\det(B)$, so $\det(A^m) = \det(A)^m$
Inverse	$\det\left(A^{-1}\right) = 1/\det(A)$
Sum	no rule: $\det(A+B) \neq \det(A) + \det(B)$ in general

EXAMPLE 5

Entries times cofactors from a DIFFERENT row

$$A = \begin{bmatrix} 3 & 2 & -1 \\ 1 & 6 & 3 \\ 2 & -4 & 0 \end{bmatrix} \qquad \begin{matrix} C_{11} = 12 & C_{12} = 6 & C_{13} = -16 \\ C_{21} = 4 & C_{22} = 2 & C_{23} = 16 \\ C_{31} = 12 & C_{32} = -10 & C_{33} = 16 \end{matrix}$$

- Row 1 with its OWN cofactors: $3(12) + 2(6) + (-1)(-16) = 64 = \det(A)$.
- Row 1 with the cofactors of ROW 2: $3(4) + 2(2) + (-1)(16) = 0$.

> WHY ZERO? Replace row 2 of A by a copy of row 1. The new matrix has two equal rows, so its determinant is 0 — and expanding it along row 2 gives exactly $3C_{21} + 2C_{22} + (-1)C_{23}$, since the cofactors of row 2 never look at row 2.

The matrix of cofactors and the adjoint

If A is $n \times n$ and C_{ij} is the cofactor of a_{ij}, then $[C_{ij}]$ is the MATRIX OF COFACTORS of A. Its TRANSPOSE is the ADJOINT of A, written $\text{adj}(A)$.

$$\text{adj}(A) = \begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix}$$

Example 6 — the matrix of Example 5

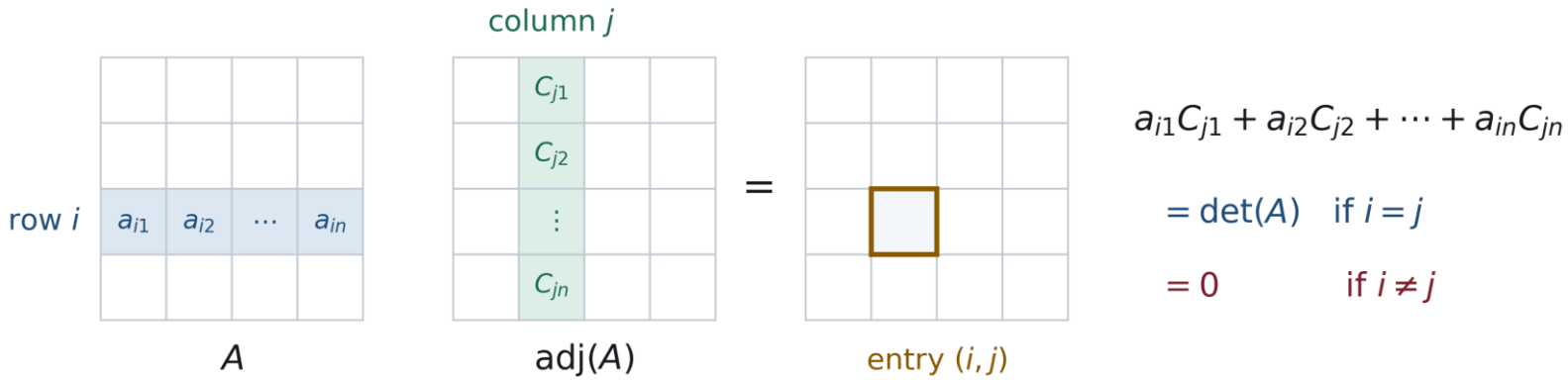
$$[C_{ij}] = \begin{bmatrix} 12 & 6 & -16 \\ 4 & 2 & 16 \\ 12 & -10 & 16 \end{bmatrix}, \qquad \text{adj}(A) = \begin{bmatrix} 12 & 4 & 12 \\ 6 & 2 & -10 \\ -16 & 16 & 16 \end{bmatrix}$$

- For 2×2: $\text{adj}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ — swap the diagonal entries, negate the other two.

THEOREM 2.3.6

The inverse from the adjoint

> If A is invertible, then $A^{-1} = \frac{1}{\det(A)}\operatorname{adj}(A)$.



Entry (i, j) of $A\operatorname{adj}(A)$ is $a_{i1}C_{j1} + \cdots + a_{in}C_{jn}$: a cofactor expansion of $\det(A)$ when $i = j$, and 0 when $i \neq j$ (Example 5). Hence

$$A\operatorname{adj}(A) = \det(A)\, I \qquad \Rightarrow \qquad A\left[\frac{1}{\det(A)}\operatorname{adj}(A)\right] = I$$

Since $\det(A) \neq 0$ the division is allowed, and a one-sided inverse is the inverse (Theorem 1.6.3). ■

EXAMPLE 7

Using the adjoint to find A^{-1}

- For the matrix A of Examples 5 and 6, $\det(A) = 64$ came free with the first-row cofactors:

$$A^{-1} = \frac{1}{64}\operatorname{adj}(A) = \frac{1}{64}\begin{bmatrix} 12 & 4 & 12 \\ 6 & 2 & -10 \\ -16 & 16 & 16 \end{bmatrix} = \frac{1}{32}\begin{bmatrix} 6 & 2 & 6 \\ 3 & 1 & -5 \\ -8 & 8 & 8 \end{bmatrix}$$

- Every entry of $\operatorname{adj}(A)$ is even, so the factor $1/64$ simplifies to $1/32$ — keep the fraction outside the matrix.

- Check one entry of $AA^{-1} = I$ — row 1 of A times column 1 of A^{-1}: $1/32\,(3 \cdot 6 + 2 \cdot 3 + (-1)(-8)) = 32/32 = 1$.

EXAMPLE

The adjoint method from start to finish

$$A = \begin{bmatrix} -1 & 3 & 2 \\ 0 & -2 & 1 \\ 1 & 0 & -2 \end{bmatrix}$$

Step 1 — all nine cofactors, with signs from the checkerboard

$$C_{11} = +\begin{vmatrix} -2 & 1 \\ 0 & -2 \end{vmatrix} = 4, \qquad C_{12} = -\begin{vmatrix} 0 & 1 \\ 1 & -2 \end{vmatrix} = 1, \qquad C_{13} = +\begin{vmatrix} 0 & -2 \\ 1 & 0 \end{vmatrix} = 2$$

$$C_{21} = -\begin{vmatrix} 3 & 2 \\ 0 & -2 \end{vmatrix} = 6, \qquad C_{22} = +\begin{vmatrix} -1 & 2 \\ 1 & -2 \end{vmatrix} = 0, \qquad C_{23} = -\begin{vmatrix} -1 & 3 \\ 1 & 0 \end{vmatrix} = 3$$

$$C_{31} = +\begin{vmatrix} 3 & 2 \\ -2 & 1 \end{vmatrix} = 7, \qquad C_{32} = -\begin{vmatrix} -1 & 2 \\ 0 & 1 \end{vmatrix} = 1, \qquad C_{33} = +\begin{vmatrix} -1 & 3 \\ 0 & -2 \end{vmatrix} = 2$$

EXAMPLE (continued)

The adjoint, a free check, and the inverse

Step 2 — transpose the matrix of cofactors

$$[C_{ij}] = \begin{bmatrix} 4 & 1 & 2 \\ 6 & 0 & 3 \\ 7 & 1 & 2 \end{bmatrix} \quad \Rightarrow \quad \operatorname{adj}(A) = \begin{bmatrix} 4 & 6 & 7 \\ 1 & 0 & 1 \\ 2 & 3 & 2 \end{bmatrix}$$

Step 3 — a free check, which also gives $\det(A)$

$$A\operatorname{adj}(A) = \begin{bmatrix} -1 & 3 & 2 \\ 0 & -2 & 1 \\ 1 & 0 & -2 \end{bmatrix} \begin{bmatrix} 4 & 6 & 7 \\ 1 & 0 & 1 \\ 2 & 3 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} = 3I \quad \Rightarrow \quad \det(A) = 3$$

Step 4 — divide

$$A^{-1} = \frac{1}{3}\operatorname{adj}(A) = \begin{bmatrix} 4/3 & 2 & 7/3 \\ 1/3 & 0 & 1/3 \\ 2/3 & 1 & 2/3 \end{bmatrix}$$

THEOREM 2.3.7

Cramer's rule

If $Ax = b$ is a system of n linear equations in n unknowns such that $\det(A) \neq 0$, then the system has a unique solution, given by

$$x_1 = \frac{\det(A_1)}{\det(A)}, \qquad x_2 = \frac{\det(A_2)}{\det(A)}, \qquad \dots, \qquad x_n = \frac{\det(A_n)}{\det(A)}$$

- where A_j is the matrix obtained by replacing the j-th column of A by b:



The denominator is always $\det(A)$; only the numerator changes. For $n = 2$ these are exactly the fractions of the motivation slide of Section 2.1.

PROOF

Why Cramer's rule works

Since $\det(A) \neq 0$, Theorems 1.6.2 and 2.3.6 give

$$x = A^{-1}b = \frac{1}{\det(A)}\operatorname{adj}(A)\, b = \frac{1}{\det(A)}\begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix}\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

so the j-th entry of x is

$$x_j = \frac{b_1 C_{1j} + b_2 C_{2j} + \cdots + b_n C_{nj}}{\det(A)}$$

The numerator is the cofactor expansion of $\det(A_j)$ along column j: A_j differs from A only in column j, so the cofactors $C_{1j}, \ldots, C_{nj}$ of that column are the same for both matrices. ■

EXAMPLE 8

Using Cramer's rule

$$\begin{aligned} x_1 + 2x_3 &= 6 \\ -3x_1 + 4x_2 + 6x_3 &= 30 \\ -x_1 - 2x_2 + 3x_3 &= 8 \end{aligned} \qquad A = \begin{bmatrix} 1 & 0 & 2 \\ -3 & 4 & 6 \\ -1 & -2 & 3 \end{bmatrix}, \quad \det(A) = 44$$

$$A_1 = \begin{bmatrix} 6 & 0 & 2 \\ 30 & 4 & 6 \\ 8 & -2 & 3 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} 1 & 6 & 2 \\ -3 & 30 & 6 \\ -1 & 8 & 3 \end{bmatrix}, \qquad A_3 = \begin{bmatrix} 1 & 0 & 6 \\ -3 & 4 & 30 \\ -1 & -2 & 8 \end{bmatrix}$$

$$x_1 = \frac{\det(A_1)}{\det(A)} = \frac{-40}{44} = -\frac{10}{11}, \qquad x_2 = \frac{72}{44} = \frac{18}{11}, \qquad x_3 = \frac{152}{44} = \frac{38}{11}$$

EXAMPLE

Cramer's rule with a parameter

$$\begin{aligned} 3s\,x_1 - 2x_2 &= 4 \\ -6x_1 + s\,x_2 &= 1 \end{aligned} \qquad \det(A) = \begin{vmatrix} 3s & -2 \\ -6 & s \end{vmatrix} = 3s^2 - 12 = 3(s+2)(s-2)$$

- So there is a unique solution exactly when $s \neq \pm 2$, and then

$$x_1 = \frac{\begin{vmatrix} 4 & -2 \\ 1 & s \end{vmatrix}}{3(s+2)(s-2)} = \frac{4s+2}{3(s+2)(s-2)}, \qquad x_2 = \frac{\begin{vmatrix} 3s & 4 \\ -6 & 1 \end{vmatrix}}{3(s+2)(s-2)} = \frac{s+8}{(s+2)(s-2)}$$

> This is where Cramer's rule earns its keep: a FORMULA for the solution in terms of s, with no row reduction of letters. (Lay, §3.3, Example 2.)

COMPARISON

Four ways to solve a square linear system

	Applies to	What you get	Best when
Gaussian elimination	any system	REF, then back-substitute	working by hand
Gauss–Jordan	any system	RREF: read it off	free variables appear
$x = A^{-1}b$	square, $\det(A) \neq 0$	the unique solution	same A, many b
Cramer's rule	square, $\det(A) \neq 0$	one unknown at a time	small n; formulas

- For $n > 3$, Gauss–Jordan elimination is far cheaper than Cramer's rule or the adjoint: computing ONE determinant costs about as much as solving the whole system by elimination.

The value of $A^{-1} = \frac{1}{\det(A)}\operatorname{adj}(A)$ and of Cramer's rule is the FORMULA: it shows how the solution depends on the data.

THEOREM 2.3.8

Equivalent Statements — the list grows to seven

- If A is an $n \times n$ matrix, the following are equivalent — all true, or all false.

(a) A is invertible.

(b) $Ax = 0$ has only the trivial solution.

(c) The reduced row echelon form of A is I_n.

(d) A is expressible as a product of elementary matrices.

(e) $Ax = b$ is consistent for every $n \times 1$ matrix b.

(f) $Ax = b$ has exactly one solution for every $n \times 1$ matrix b.

(g) $\det(A) \neq 0$.

> Read it in the negative too: $\det(A) = 0$ exactly when $Ax = 0$ has a nontrivial solution, exactly when $\mathrm{RREF}(A)$ has a row of zeros, exactly when $Ax = b$ is inconsistent for at least one b.

PROMISE KEPT · OPTIONAL

Proving Theorem 1.7.1 (c) and (d)

- Section 1.7 stated these without proof; determinants were the missing tool.

(c) A triangular matrix is invertible exactly when its diagonal entries are all nonzero

$\det(A) = a_{11}a_{22}\cdots a_{nn}$ by Theorem 2.1.2, and a product is nonzero exactly when every factor is. Now apply Theorem 2.3.3.

(d) The inverse of an invertible upper triangular matrix is upper triangular

By Theorem 2.3.6 it is enough that $C_{ij} = 0$ whenever $i < j$. Deleting row i and column j of A leaves an upper triangular matrix whose i-th row starts with at least i zeros — a zero on its diagonal — so $M_{ij} = 0$. Then $\left[C_{ij}\right]$ is lower triangular and $\mathrm{adj}(A)$, its transpose, is upper triangular. The lower triangular case is the same.

Mistakes to avoid

- Forgetting the checkerboard: $C_{ij} = (-1)^{i+j} M_{ij}$, not M_{ij}.
- Taking the entries from one row and the cofactors from another.
- Using the arrow technique on a 4×4 determinant.
- Treating a row replacement as if it multiplied the determinant.
- Losing track of interchanges: each one flips the sign, and two cancel.
- Writing $\det(kA) = k\det(A)$. It is $k^n \det(A)$.
- Writing $\det(A + B) = \det(A) + \det(B)$. There is no such rule.
- Forgetting to TRANSPOSE the matrix of cofactors when forming $\text{adj}(A)$.
- Using Cramer's rule when $\det(A) = 0$, or when the system is not square.

SUMMARY

Unit 2 in one slide

2.1 — the determinant, by cofactor expansion

- Minors M_{ij} and cofactors $C_{ij} = (-1)^{i+j} M_{ij}$; the expansion along ANY row or column gives $\det(A)$. Triangular: multiply the diagonal. Arrows: 2×2 and 3×3 only.

2.2 — the determinant, by row reduction

- Scaling a row by k multiplies the determinant by k; an interchange flips its sign; a replacement changes nothing — for columns as well as rows. Reduce to triangular form and keep score.

2.3 — properties and formulas

- $\det(kA) = k^n \det(A)$, $\det(AB) = \det(A)\det(B)$, $\det\left(A^{-1}\right) = 1/\det(A)$, and no rule for sums. A is invertible exactly when $\det(A) \neq 0$; then $A^{-1} = \frac{1}{\det(A)} \operatorname{adj}(A)$ and $x_j = \det\left(A_j\right)/\det(A)$.