2 Determinants

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2.1 Determinants by Cofactor Expansion

The matrix $\tilde{A}_{ij}$

If $n \geq 2$, $A \in M_{n\times n}$, then $\tilde{A}_{ij} \in M_{(n-1)\times(n-1)}$ is the matrix found by deleting the i-th row and j-th column of A.

Example

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad \tilde{A}_{11} = [d]$$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}, \quad \tilde{A}_{22} = \begin{bmatrix} 1 & 3 \\ 7 & 9 \end{bmatrix}$$

Definition of Determinant $|A|$ of a matrix $A \in M_{n\times n}$

By induction on $n \geq 1$:

- $n = 1 \Rightarrow |A| = A_{11}$
- $n \geq 2 \Rightarrow |A| = A_{11}|\tilde{A}_{11}| - A_{12}|\tilde{A}_{12}| + \cdots + (-1)^{n+1} A_{1n}|\tilde{A}_{1n}|$

Note $\det(A)$ is also used to denote the determinant of A.

Example Computing Determinants

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = 1\begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} - 2\begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} + 3\begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix}$$

$$= 1(-3) - 2(-6) + 3(-3) = -3 + 12 - 9 = 0$$

Definition — Minors and Cofactors

- $M_{ij} = |\tilde{A}_{ij}|$ is called the **minor** of A_{ij}.
- $C_{ij} = (-1)^{i+j}|\tilde{A}_{ij}| = (-1)^{i+j}M_{ij}$ is called the **cofactor** of A_{ij}.

Definition — The matrices $M, C, \mathrm{adj}(A)$

$$M = \begin{bmatrix} M_{11} & M_{12} & \cdots & M_{1n} \\ M_{21} & M_{22} & \cdots & M_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ M_{n1} & M_{n2} & \cdots & M_{nn} \end{bmatrix} = \begin{bmatrix} |\tilde{A}_{11}| & |\tilde{A}_{12}| & \cdots & |\tilde{A}_{1n}| \\ |\tilde{A}_{21}| & |\tilde{A}_{22}| & \cdots & |\tilde{A}_{2n}| \\ \vdots & \vdots & \ddots & \vdots \\ |\tilde{A}_{n1}| & |\tilde{A}_{n2}| & \cdots & |\tilde{A}_{nn}| \end{bmatrix}$$

$$C = \begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & \cdots & C_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ C_{n1} & C_{n2} & \cdots & C_{nn} \end{bmatrix} = \begin{bmatrix} M_{11} & -M_{12} & \cdots & (-1)^{1+n}M_{1n} \\ -M_{21} & M_{22} & \cdots & (-1)^{2+n}M_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ (-1)^{n+1}M_{n1} & (-1)^{n+2}M_{n2} & \cdots & M_{nn} \end{bmatrix}, \quad \mathrm{adj}(A) = C^T$$

Example Finding Minors and Cofactors

Let $A = \begin{bmatrix} 1 & 3 & -1 \\ -2 & 5 & 4 \\ 6 & 2 & 0 \end{bmatrix}$.

- The minor of A_{23} is $M_{23} = \begin{vmatrix} 1 & 3 \\ 6 & 2 \end{vmatrix} = -16$.
- The cofactor of A_{23} is $C_{23} = (-1)^5 M_{23} = -(-16) = 16$.

Example Adjoint of a 2 by 2 matrix

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then

$$M = \begin{bmatrix} d & c \\ b & a \end{bmatrix},$$

$$C = \begin{bmatrix} d & -c \\ -b & a \end{bmatrix},$$

$$adj(A) = C^T = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Theorem $|A|$ can be computed using any row or column

The number obtained by multiplying the entries in any row or column of A by the corresponding cofactors and adding the resulting products is always $|A|$. So

- Using row i: $|A| = A_{i1}C_{i1} + A_{i2}C_{i2} + \cdots + A_{in}C_{in}.$
- Using column j: $|A| = A_{1j}C_{1j} + A_{2j}C_{2j} + \cdots + A_{nj}C_{nj}.$

Example Determinant of 2 by 2 along all possible rows and columns

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}C_{11} + a_{12}C_{12} = a_{11}a_{22} - a_{12}a_{21}$$

$$= a_{21}C_{21} + a_{22}C_{22} = -a_{21}a_{12} + a_{22}a_{11}$$

$$= a_{11}C_{11} + a_{21}C_{21} = a_{11}a_{22} - a_{21}a_{12}$$

$$= a_{12}C_{12} + a_{22}C_{22} = -a_{12}a_{21} + a_{22}a_{11}$$

Example Computing determinants along different rows and columns

Computing the determinant of $A = \begin{bmatrix} 2 & 1 & 1 \\ 3 & 4 & -1 \\ 1 & 5 & 1 \end{bmatrix}$

Along 1st row: $\det(A) = 2\begin{vmatrix} 4 & -1 \\ 5 & 1 \end{vmatrix} - 1\begin{vmatrix} 3 & -1 \\ 1 & 1 \end{vmatrix} + 1\begin{vmatrix} 3 & 4 \\ 1 & 5 \end{vmatrix} = 2(9) - 1(4) + 1(11) = 25$

Along 1st column: $\det(A) = 2\begin{vmatrix} 4 & -1 \\ 5 & 1 \end{vmatrix} - 3\begin{vmatrix} 1 & 1 \\ 5 & 1 \end{vmatrix} + 1\begin{vmatrix} 1 & 1 \\ 4 & -1 \end{vmatrix} = 2(9) - 3(-4) + 1(-5) = 25$

Along 3rd column: $\det(A) = 1\begin{vmatrix} 3 & 4 \\ 1 & 5 \end{vmatrix} - (-1)\begin{vmatrix} 2 & 1 \\ 1 & 5 \end{vmatrix} + 1\begin{vmatrix} 2 & 1 \\ 3 & 4 \end{vmatrix} = 1(11) + 1(9) + 1(5) = 25$

Theorem Determinant of a Triangular Matrices

If A is triangular, then $|A| = A_{11}A_{22}\cdots A_{nn}$.

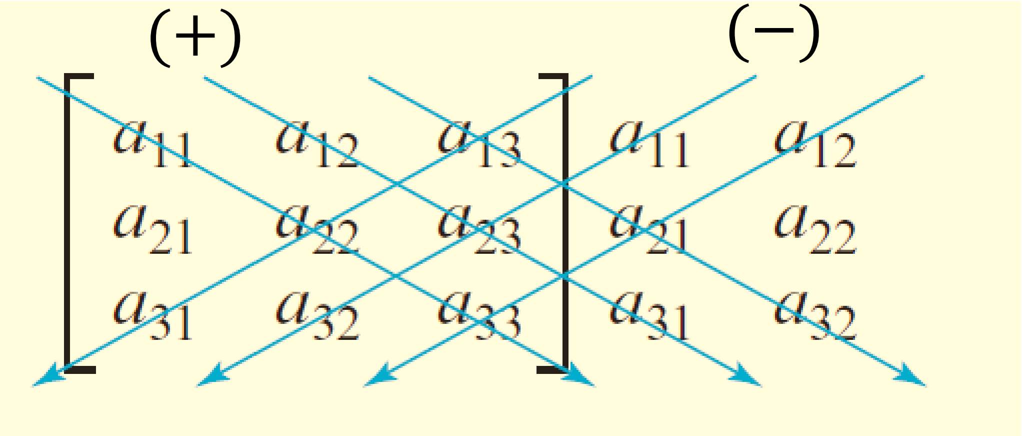
Proof Evaluating repeatedly using the first column.

Example Determinant of Triangular Matrices

$$\begin{vmatrix} -1 & 0 & 0 & 0 \\ 3 & 4 & 0 & 0 \\ 6 & 0 & \frac{1}{2} & 6 \\ 1 & 1 & 1 & 3 \end{vmatrix} = (-1)(4)(\frac{1}{2})(3) = -6$$

$$\begin{vmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{vmatrix} = (2)(3)(4) = 24$$

Technique for determinant of 3 by 3



Example Computing 3 by 3 determinant using arrow technique

Computing the determinant of $A = \begin{bmatrix} 2 & 1 & 1 \\ 3 & 4 & -1 \\ 1 & 5 & 1 \end{bmatrix}$ using arrow technique

2	1	1	2	1
3	4	−1	3	4
1	5	1	1	5

$$|A| = (2 \cdot 4 \cdot 1) + (1 \cdot (-1) \cdot 1) + (1 \cdot 3 \cdot 5) - [(1 \cdot 4 \cdot 1) + (2 \cdot (-1) \cdot 5) + (1 \cdot 3 \cdot 1)]$$
$$= (8) + (-1) + (15) - [(4) + (-1) + (30)] = 22 - (-3) = 25$$

2.2 Evaluating Determinants by Row Reduction

Theorem A class of zero determinant matrices

If $A \in M_{n\times n}$ has a zero row or a zero column, then $|A| = 0$.

Proof Evaluating $|A|$ using that zero row or column and assuming its cofactors are $C_1, C_2, \cdots, C_n$, we have:

$$|A| = 0C_1 + 0C_2 + \cdots + 0C_n = 0$$

Theorem Effect of Elementary Row Operations on Determinants

Let $A \in M_{n\times n}$.

1. If $A \xrightarrow{M_i^c} B$, then $|B| = c|A|$.
2. If $A \xrightarrow{I_{ij}} B$, then $|B| = -|A|$.
3. If $A \xrightarrow{A_{ij}^c} B$, then $|B| = |A|$.

Note We can simplify a matrix $A \in M_{n\times n}$ using elementary row operations then compute $|A|$.

Example evaluating determinants using row reduction

If $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 10 \end{bmatrix}$, Find $|A|$.

Sol.

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 10 \end{bmatrix} \underset{A_{13}^{-7}}{\overset{A_{12}^{-4}}{\rightarrow}} \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & -6 & -11 \end{bmatrix} \overset{A_{23}^{-2}}{\rightarrow} \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & 0 & 1 \end{bmatrix}$$

Since we have a triangular matrix and we only used operation (3),

$$|A| = (1)(-3)(1) = -3$$

Corollary Determinants of Elementary Matrices

1. If $I \xrightarrow{M_i^c} E, c \neq 0$, then $|E| = c.$
2. If $I \xrightarrow{I_{ij}} E$, then $|E| = -1.$
3. If $I \xrightarrow{A_{ij}^c} E$, then $|E| = 1.$

Note An elementary matrix has nonzero determinant.

Lemma Determinants of a product with an Elementary Matrix

1. If E is an elementary matrix, then $|EA| = |E||A|$.
2. If $E_1, E_2, \cdots, E_k$ are elementary matrices, then $|E_1 E_2 \cdots E_k A| = |E_1||E_2| \cdots |E_k||A|$

Lemma Determinant test for Invertibility

A square matrix A is invertible if and only if $|A| \neq 0.$

Proof By reduction $rref(A) = E_k\ E_{k-1} \cdots E_1 A.$ Taking determinants

$$\Rightarrow |rref(A)| = |E_k\ E_{k-1} \cdots E_1 A| = |E_k||E_{k-1}| \cdots |E_1||A|$$

Now A is invertible if and only if $rref(A) = I \Longrightarrow |rref(A)| = 1 \Longrightarrow |A| \neq 0.$ On the other hand, if A is not invertible then $rref(A)$ has a zero row so $|rref(A)| = 0$ and since $|E_k||E_{k-1}| \cdots |E_1| \neq 0$, we must have $|A| = 0.$

Example Checking Invertibility Using Determinants

Note that $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ is not invertible since we saw $\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = 0$

2.3 Properties of Determinants; Cramer's Rule

Theorem Effect of Matrix Operations on the Determinant

For any $A, B \in M_{n\times n}$, and any $c \in \mathbb{R}$:

1. $|A+B| \neq |A| + |B|$ in general.
2. $|cA| = c^n|A|$.
3. $|A^T| = |A|$.
4. $|AB| = |A||B|$.
5. If A is invertible, $|A^{-1}| = \frac{1}{|A|}$.

Example Using Properties to Find Determinants

$$A = \begin{bmatrix} 10 & -20 & 40 \\ 30 & 0 & 50 \\ -20 & -30 & 10 \end{bmatrix}, \text{ if } \begin{vmatrix} 1 & -2 & 4 \\ 3 & 0 & 5 \\ -2 & -3 & 1 \end{vmatrix} = 5, \text{ find } |A|$$

Sol.

$$A = 10\begin{bmatrix} 1 & -2 & 4 \\ 3 & 0 & 5 \\ -2 & -3 & 1 \end{bmatrix} \Rightarrow |A| = 10^3 \begin{vmatrix} 1 & -2 & 4 \\ 3 & 0 & 5 \\ -2 & -3 & 1 \end{vmatrix} = (1000)(5) = 5000$$

Corollary Classes of Matrices With Zero Determinants

A matrix will have zero determinant in each of the following cases:

1. Two rows or columns are equal.
2. Two rows or columns are multiples of each other.

Lemma An Additive Property of Determinants

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21}+b_{21} & a_{22}+b_{22} & a_{23}+b_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ b_{21} & b_{22} & b_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

Note: This property is also valid for any rows or columns other than the second row.

Example Verifying the Property

$$|A| = \begin{vmatrix} 1 & -2 & 2 \\ 0 & 3 & 2 \\ 1 & 0 & 1 \end{vmatrix} \overset{?}{=} \begin{vmatrix} 1 & -2 & 2 \\ -1 & 1 & 2 \\ 1 & 0 & 1 \end{vmatrix} + \begin{vmatrix} 1 & -2 & 2 \\ 1 & 2 & 0 \\ 1 & 0 & 1 \end{vmatrix} = |B| + |C|$$

Pf.

$$|A| = 0(-1)^{2+1}\begin{vmatrix} -2 & 2 \\ 0 & 1 \end{vmatrix} + 3(-1)^{2+2}\begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix} + 2(-1)^{2+3}\begin{vmatrix} 1 & -2 \\ 1 & 0 \end{vmatrix}$$

$$|B| = -1(-1)^{2+1}\begin{vmatrix} -2 & 2 \\ 0 & 1 \end{vmatrix} + 1(-1)^{2+2}\begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix} + 2(-1)^{2+3}\begin{vmatrix} 1 & -2 \\ 1 & 0 \end{vmatrix}$$

$$|C| = 1(-1)^{2+1}\begin{vmatrix} -2 & 2 \\ 0 & 1 \end{vmatrix} + 2(-1)^{2+2}\begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix} + 0(-1)^{2+3}\begin{vmatrix} 1 & -2 \\ 1 & 0 \end{vmatrix}$$

Theorem Equivalent Statements for a Square Matrix

Let $A \in M_{n\times n}$. The following are equivalent statements.

1. A is invertible.
2. $Ax = 0$ has only the trivial solution.
3. $RREF(A) = I$.
4. A is a product of elementary matrices.
5. $Ax = b$ is consistent for every $n \times 1$ matrix b.
6. $Ax = b$ has exactly one solution for every $n \times 1$ matrix b.
7. $|A| \neq 0$.

Example Checking Square System Solutions

Which of the following system has a unique solution?

(a)
$$\begin{array}{rcrcrcr} & & 2x_2 & - & x_3 & = & -1 \\ 3x_1 & - & 2x_2 & + & x_3 & = & 4 \\ 3x_1 & + & 2x_2 & - & x_3 & = & -4 \end{array}$$

(b)
$$\begin{array}{rcrcrcr} & & 2x_2 & - & x_3 & = & -1 \\ 3x_1 & - & 2x_2 & + & x_3 & = & 4 \\ 3x_1 & + & 2x_2 & + & x_3 & = & -4 \end{array}$$

Sol:

(a) $A\mathbf{x} = \mathbf{b}$, the coefficient matrix is $A = \begin{bmatrix} 0 & 2 & -1 \\ 3 & -2 & 1 \\ 3 & 2 & -1 \end{bmatrix}$ has $|A| = 0 \Rightarrow$ NOT unique solution.

(b) $B\mathbf{x} = \mathbf{b}$, the coefficient matrix is $B = \begin{bmatrix} 0 & 2 & -1 \\ 3 & -2 & 1 \\ 3 & 2 & 1 \end{bmatrix}$ has $|B| = -12 \neq 0 \Rightarrow$ Unique solution.

Theorem The inverse of a matrix expressed by its adjoint matrix

If A is an $n \times n$ invertible matrix, then $A^{-1} = \dfrac{1}{\det(A)} \operatorname{adj}(A)$

Proof: Consider the product $A[\operatorname{adj}(A)]$

$$= \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} C_{11} & \cdots & C_{j1} & \cdots & C_{n1} \\ C_{12} & \cdots & C_{j2} & \cdots & C_{n2} \\ \vdots & & \vdots & & \vdots \\ C_{1n} & \cdots & C_{jn} & \cdots & C_{nn} \end{bmatrix} = \begin{bmatrix} \det(A) & 0 & \cdots & 0 \\ 0 & \det(A) & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & \det(A) \end{bmatrix}$$

The entry at the position (i, j) of $A[\operatorname{adj}(A)]$

$$a_{i1}C_{j1} + a_{i2}C_{j2} + \cdots + a_{in}C_{jn} = \begin{cases} \det(A) & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

Consider a matrix B similar to matrix A except that the j-th row is replaced by the i-th row of matrix A

$$\Rightarrow \det(B) = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = 0$$

※ Since there are two identical rows in B, according to a previous Theorem, $\det(B)$ should be zero

Perform the cofactor expansion along the j-th row of matrix B

$$\Rightarrow \det(B) = a_{i1}C_{j1} + a_{i2}C_{j2} + \cdots + a_{in}C_{jn} = 0$$

(Note that C_{j1}, C_{j2},…, and C_{jn} are still the cofactors for the entries of the j-th row)

$$\Rightarrow A[\operatorname{adj}(A)] = \det(A)I \Rightarrow A\left[\underbrace{\frac{1}{\det(A)}\operatorname{adj}(A)}_{A^{-1}}\right] = I$$

Example Inverse for 2 by 2 matrix

For any 2×2 matrix, its inverse can be calculated as follows

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \Rightarrow \det(A) = ad - bc,$$

$$\operatorname{adj}(A) = \begin{bmatrix} C_{11} & C_{21} \\ C_{12} & C_{22} \end{bmatrix} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$\Rightarrow A^{-1} = \frac{1}{\det(A)} \operatorname{adj}(A) = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Example The Inverse Using the Adjoint

$$A=\begin{bmatrix}-1 & 3 & 2\\ 0 & -2 & 1\\ 1 & 0 & -2\end{bmatrix}$$

(a) Find the adjoint matrix of A

(b) Use the adjoint matrix of A to find A^{-1}

Sol.

$$\because \quad C_{ij}=(-1)^{i+j}M_{ij}$$

$$\Rightarrow C_{11}=+\begin{vmatrix}-2 & 1\\ 0 & -2\end{vmatrix}=4,\quad C_{12}=-\begin{vmatrix}0 & 1\\ 1 & -2\end{vmatrix}=1,\quad C_{13}=+\begin{vmatrix}0 & -2\\ 1 & 0\end{vmatrix}=2,$$

$$C_{21}=-\begin{vmatrix}3 & 2\\ 0 & -2\end{vmatrix}=6,\quad C_{22}=+\begin{vmatrix}-1 & 2\\ 1 & -2\end{vmatrix}=0,\quad C_{23}=-\begin{vmatrix}-1 & 3\\ 1 & 0\end{vmatrix}=3,$$

$$C_{31}=+\begin{vmatrix}3 & 2\\ -2 & 1\end{vmatrix}=7,\quad C_{32}=-\begin{vmatrix}-1 & 2\\ 0 & 1\end{vmatrix}=1,\quad C_{33}=+\begin{vmatrix}-1 & 3\\ 0 & -2\end{vmatrix}=2.$$

$\Rightarrow$ cofactor matrix of A $\Rightarrow$ adjoint matrix of A

$$[C_{ij}] = \begin{bmatrix} 4 & 1 & 2 \\ 6 & 0 & 3 \\ 7 & 1 & 2 \end{bmatrix} \qquad \text{adj}(A) = [C_{ij}]^T = \begin{bmatrix} 4 & 6 & 7 \\ 1 & 0 & 1 \\ 2 & 3 & 2 \end{bmatrix}$$

$\Rightarrow$ inverse matrix of A

$$A^{-1} = \frac{1}{\det(A)}\text{adj}(A) \quad (\det(A) = 3)$$

$$= \tfrac{1}{3}\begin{bmatrix} 4 & 6 & 7 \\ 1 & 0 & 1 \\ 2 & 3 & 2 \end{bmatrix} = \begin{bmatrix} \frac{4}{3} & 2 & \frac{7}{3} \\ \frac{1}{3} & 0 & \frac{1}{3} \\ \frac{2}{3} & 1 & \frac{2}{3} \end{bmatrix}$$

※ The computational effort of this method to derive the inverse of a matrix is higher than that of the G.-J. E. (especially to compute the cofactor matrix for a higher-order square matrix)

※ However, for computers, it is easier to implement this method than the G.-J. E. since it is not necessary to judge which row operation should be used and the only thing needed to do is to calculate determinants of matrices

- Check: $AA^{-1} = I$

Theorem Cramer's Rule

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n \end{aligned} \quad \Rightarrow A\mathbf{x} = \mathbf{b}$$

$A^{(i)}$ represents the i-th column vector in A

where $A = \left[a_{ij}\right]_{n\times n} = \left[A^{(1)}\ A^{(2)}\ \cdots\ A^{(n)}\right]$, $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$

Suppose this system has a unique solution, i.e. $\det(A) \neq 0$. Then

By defining $A_j = \left[A^{(1)}\ A^{(2)}\ \cdots\ A^{(j-1)}\ \mathbf{b}\ A^{(j+1)}\ \cdots\ A^{(n)} \right]$

$$= \begin{bmatrix} a_{11} & \cdots & a_{1(j-1)} & b_1 & a_{1(j+1)} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2(j-1)} & b_2 & a_{2(j+1)} & \cdots & a_{2n} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ a_{n1} & \cdots & a_{n(j-1)} & b_n & a_{n(j+1)} & \cdots & a_{nn} \end{bmatrix}$$

(i.e., $\det(A_j) = b_1 C_{1j} + b_2 C_{2j} + \cdots + b_n C_{nj}$)

$$\Rightarrow x_j = \frac{\det(A_j)}{\det(A)},\ j = 1, 2, \cdots, n$$

Proof

$$A\mathbf{x} = \mathbf{b} \quad (\det(A) \neq 0)$$

$$\Rightarrow \mathbf{x} = A^{-1}\mathbf{b} = \frac{1}{\det(A)}\operatorname{adj}(A)\mathbf{b}$$ (according to the previous Thm.)

$$= \frac{1}{\det(A)}\begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix}\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

$$= \frac{1}{\det(A)}\begin{bmatrix} b_1C_{11} + b_2C_{21} + \cdots + b_nC_{n1} \\ b_1C_{12} + b_2C_{22} + \cdots + b_nC_{n2} \\ \vdots \\ b_1C_{1n} + b_2C_{2n} + \cdots + b_nC_{nn} \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \frac{1}{\det(A)} \begin{bmatrix} b_1C_{11} + b_2C_{21} + \cdots + b_nC_{n1} \\ b_1C_{12} + b_2C_{22} + \cdots + b_nC_{n2} \\ \vdots \\ b_1C_{1n} + b_2C_{2n} + \cdots + b_nC_{nn} \end{bmatrix}$$

$$= \begin{bmatrix} \det(A_1)/\det(A) \\ \det(A_2)/\det(A) \\ \vdots \\ \det(A_n)/\det(A) \end{bmatrix}$$

(It is already derived that $\det(A_j) = b_1C_{1j} + b_2C_{2j} + \cdots + b_nC_{nj}$)

$$\Rightarrow x_j = \frac{\det(A_j)}{\det(A)}, \; j = 1, 2, \cdots, n$$

Example Inverse Soving Linear Systems by Cramer's Rule

Use Cramer's rule to solve the system of linear equation

$$\begin{array}{rcrcrcr} -x & + & 2y & - & 3z & = & 1 \\ 2x & & & + & z & = & 0 \\ 3x & - & 4y & + & 4z & = & 2 \end{array}$$

Sol.

$$\det(A) = \begin{vmatrix} -1 & 2 & -3 \\ 2 & 0 & 1 \\ 3 & -4 & 4 \end{vmatrix} = 10 \qquad \det(A_1) = \begin{vmatrix} 1 & 2 & -3 \\ 0 & 0 & 1 \\ 2 & -4 & 4 \end{vmatrix} = 8$$

$$\det(A_2) = \begin{vmatrix} -1 & 1 & -3 \\ 2 & 0 & 1 \\ 3 & 2 & 4 \end{vmatrix} = -15, \qquad \det(A_3) = \begin{vmatrix} -1 & 2 & 1 \\ 2 & 0 & 0 \\ 3 & -4 & 2 \end{vmatrix} = -16$$

$$x = \frac{\det(A_1)}{\det(A)} = \frac{4}{5} \qquad y = \frac{\det(A_2)}{\det(A)} = \frac{-3}{2} \qquad z = \frac{\det(A_3)}{\det(A)} = \frac{-8}{5}$$