King Saud University
College of Since
Mathematical Department

Mid-Term 1/ 2017-2018
Full Marke:25. Time 1H30min
24/10/2017

Question 1[4,4]. a) Find and sketch the largest local region of the $xy-$plane for which the initial value problem

$$\begin{cases} \sqrt{4-x^2}\frac{dy}{dx} = \sqrt{9-\ln(y-1)} \\ y(0) = 2. \end{cases}$$

has a unique solution.

Solution:

$$\frac{dy}{dx} = \frac{\sqrt{9-\ln(y-1)}}{\sqrt{4-x^2}} = f(x.y)$$

f is continuous for $4-x^2>0$, $y-1>0$

$-x^2>-4$, $y>1$

$|x|<2$

$-2<x<2$

$$f(x.y) = \frac{(9-\ln(y-1))^{\frac{1}{2}}}{\sqrt{4-x^2}} \quad , \quad \frac{\partial f}{\partial y} = \frac{\frac{1}{2}(9-\ln(y-1))^{-\frac{1}{2}}\left(\frac{1}{y-1}\right)}{\sqrt{4-x^2}}$$

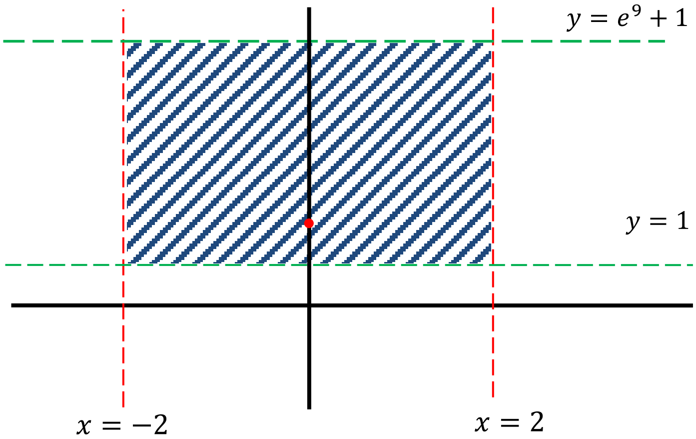
$$\frac{\partial f}{\partial y} = \frac{1}{2(y-1)\sqrt{9-\ln(y-1)}\sqrt{4-x^2}}$$

$\frac{\partial f}{\partial y}$ is continuous for $-2<x<2$. $1<y<e^9+1$

f and $\frac{\partial f}{\partial y}$ are continuous on $\{(x.y) \in \mathcal{R};\ -2<x<2\ .\ 1<y<e^9+1\}$

$(0,2) \in R_1 = \{(x.y); -2<x<2\ .\ 1<y<e^9+1\}$

R_1 is the largest local region for which the Initial Value Problem has a unique solution



b) Find the solution of the differential equation:

$$(2xy + 2xy\ln y)dx + (2+\ln y)(5-x^2)dy = 0, \quad y > e^{-1}, \ x \neq \pm\sqrt{5}.$$

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Solution: by Separable D.E.

$$(2xy + 2xy\ln y)dx + (2+\ln y)(5-x^2)dy = 0$$

$$x(2y + 2y\ln y)dx + (2+\ln y)(5-x^2)dy = 0$$

$$\frac{x}{(5-x^2)}dx = -\frac{2+\ln y}{2y+2y\ln y}dy$$

$u = 5 - x^2$

$du = -2x dx$

$-\frac{du}{2x} = dx$

$$\int \frac{x}{(5-x^2)}dx = \int -\frac{2+\ln y}{2y+2y\ln y}dy$$

$s = \ln y$

$ds = \frac{1}{y}dy$

$ds = dy$

$$-\frac{1}{2}\int \frac{1}{u}du = -\int \frac{2+\ln y}{y(2+2\ln y)}dy$$

Multiply -1 $$-\frac{1}{2}\int \frac{1}{u}du = -\frac{1}{2}\int \frac{2+s}{1+s}ds$$

$$\frac{1}{2}\int \frac{1}{u}du = \frac{1}{2}\int \frac{1+1+s}{1+s}ds$$

Multiply 2 $$\frac{1}{2}\int \frac{1}{u}du = \frac{1}{2}\int \frac{1}{1+s} + \frac{1+s}{1+s}ds$$

$$\int \frac{1}{u}du = \int \frac{1}{1+s} + 1\ ds$$

$$\ln u + c = s + \ln(1+s)$$

$$\ln(5-x^2) + c = \ln y + \ln(1+\ln y)$$

Question 2[4,4]. **a)** Solve the initial value problem

$$\begin{cases} \sqrt{y}.y' + y^{3/2} = 1, \quad y > 0 \\ y(0) = 4. \end{cases}$$

Solution: by Bernoully's Equation D.E.

$$\sqrt{y} * y' + y^{\frac{3}{2}} = 1$$

Divide by y^n $\quad y' + y = y^{-\frac{1}{2}}$

Eq1 $\quad y^{\frac{1}{2}}y' + y^{\frac{3}{2}} = 1$

Multiply $\mu(x)$ $\quad w' + w\frac{3}{2} = \frac{3}{2}$

$$y^n = y^{-\frac{1}{2}}$$

$$w = y^{1+\frac{1}{2}}$$

$$w = y^{\frac{3}{2}}$$

$$w' = \frac{3}{2}y^{\frac{1}{2}}y'$$

$$\frac{2}{3}w'y^{-\frac{1}{2}} = y'$$

$$e^x w' + e^x w\frac{3}{2} = e^x$$

$$e^{\frac{3}{2}x}w' + e^{\frac{3}{2}x}w\frac{3}{2} = e^{\frac{3}{2}x}\frac{3}{2}$$

$$\frac{d}{dx}\left(we^{\frac{3}{2}x}\right) = e^{\frac{3}{2}x}\frac{3}{2}$$

$$\int \frac{d}{dx}\left(we^{\frac{3}{2}x}\right)dx = \int e^{\frac{3}{2}x}\frac{3}{2}dx$$

$$P(x) = \frac{3}{2}$$

$$Q(x) = \frac{3}{2}$$

$$\mu(x) = e^{\int P(x)dx}$$

$$\mu(x) = e^{\int \frac{3}{2}dx}$$

$$\mu(x) = e^{\frac{3}{2}x}$$

$$we^{\frac{3}{2}x} = e^{\frac{3}{2}x} + c$$

$$w = 1 + \frac{c}{e^{\frac{3}{2}x}}$$

$$y^{\frac{3}{2}} = 1 + \frac{c}{e^{\frac{3}{2}x}}$$

$$c = (4)^{\frac{3}{2}} - 1 \;;\; c = 7$$

b) Solve the differential equation

$$(3xy + 3y - 4)dx + (x+1)^2\,dy = 0, \quad x > -1.$$

Solution: by First order linear D.E.

$$(3xy + 3y - 4)dx + (x+1)^2 dy = 0$$

$$3xy + 3y - 4 + (x+1)^2\frac{dy}{dx} = 0$$

$$3y(x+1) - 4 + (x+1)^2\frac{dy}{dx} = 0$$

$$3y(x+1) + (x+1)^2\frac{dy}{dx} = 4$$

$$(x+1)^2\frac{dy}{dx} + 3y(x+1) = 4$$

Multiply by $\mu(x)$

$$y' + y\left(\frac{3}{x+1}\right) = \frac{4}{(x+1)^2}$$

$$P(x) = \frac{3}{x+1}$$

$$Q(x) = \frac{4}{(x+1)^2}$$

$$\mu(x) = e^{3\int\frac{1}{x+1}dx}$$

$$\mu(x) = e^{3\ln x+1}$$

$$\mu(x) = e^{(\ln x+1)^3}$$

$$\mu(x) = (x+1)^3$$

$$(x+1)^3y' + y\left(\frac{3}{x+1}\right)(x+1)^3 = \frac{4}{(x+1)^2}(x+1)^3$$

$$(x+1)^3y' + y\left(\frac{3}{x+1}\right)(x+1)^3 = 4(x+1)$$

$$\frac{d}{dx}(y(x+1)^3) = 4(x+1)$$

$$(y(x+1)^3) = \int 4(x+1)\,dx$$

$$y = \frac{2x^2}{(x+1)^3} + \frac{x}{(x+1)^3} + c$$

Question 3[4]. Find the general solution of the differential equation

$$(x^3 + xy^2 - y)dx + (y^3 + x^2y - x)\ dy = 0.$$

$$(x^3 + xy^2 - y)dx = M \quad ; \quad (y^3 + x^2y - x)dy = N$$

$$\frac{\partial M}{\partial y} = 2xy - 1 \quad ; \quad \frac{\partial N}{\partial x} = 2xy - 1$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

∴D.E exact

$$\int (x^3 + xy^2 - y)dx = \frac{1}{4}x^4 + \frac{1}{2}x^2y^2 - xy$$

$$\int (y^3 + x^2y - x)dy = \frac{1}{4}y^4 + \frac{1}{2}x^2y^2 - xy$$

$$f(x.y) = \frac{1}{4}x^4 + \frac{1}{4}y^4 + \frac{1}{2}x^2y^2 - xy + c$$

$$c = \frac{x^4}{4} + \frac{y^4}{4} + \frac{x^2y^2}{2} - xy$$

Question 4[5] A thermometer reading 18^0F, is brought into a room where the temperature is 70^0F. One minute later the thermometer reading is 31^0F. Determine the temperature reading at any time t. Find the temperature reading five minutes after the thermometer is first brought into the room.

$$T = Ts + ce^{kt}$$

$$T_0 = 18\ .\ Ts = 70\ .\ c = ??\ .\ t = 0$$

$$18 = 70 + ce^0$$

$$18 - 70 = c$$

$$c = -52$$

$$T_1 = 31\ .\ Ts = 70\ .\ c = -52\ .\ t = 1\ .\ k = ??$$

$$31 = 70 - 52e^k$$

$$31 - 70 = -52e^k$$

$$\frac{-39}{-52} = e^k$$

$$\ln\frac{3}{4} = k\ \ln e$$

$$k = -0.2876$$

$$T_5 = ??\ .\ Ts = 70\ .\ c = -52\ .\ t = 5\ .\ k = -0.2876$$

$$T_5 = 70 - 52e^{5(-0.2876)}$$

$$T = 57.65$$