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| KSU logo tiff.tif |  **King Saud University**  |
|  **College of Sciences** |
|  **Department of Mathematics** |
|  **373 Math** |
|  **Final Exam** |
|  **Second Semester 1433-1434** |

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**Question 1:**

1. Let $\left(X,d\right)$ be a metric space. Show that the function $e:X×X\rightarrow R$ defined by $e\left(x,y\right)=min\left\{1,d(x,y)\right\}$ is a metric on $X$. Show that $d$ and $e$ induce the same topology.
2. Let $\left(X,d\right)$ be a metric space. Prove that if $A$ is a compact subset of $X$, then $A$ is closed and bounded. Give an example to show the converse of the statement does not hold.
3. What do we mean by the metrizability problem? Is every topological space metrizable? (Justify your answer)

**Question 2:**

1. Let $\left(X,d\right)$ be a metric space. Prove that the set $\overbar{B}\left(x,ϵ\right)=\left\{y\in X:d(x,y)\leq ϵ\right\}$ is closed, where $x\in X$ and $ϵ>0$. (This set is called the closed ball with center $x$ and radius $ϵ.)$
2. Let $τ$ be the usual topology on $R^{n}$, $n\in N$. Prove that $\left(R^{n},τ\right)$ is a metrizable.

**Question 3:**

1. Prove that in a Hausdorff space any convergent sequence has a unique limit. Give an example to show the converse of the statement does not hold.
2. Let $\left(x\_{n}\right)$ and $\left(y\_{n}\right)$ be sequences in the spaces $X$ and $Y$, respectively. Prove that the sequence $\left(\left(x\_{n},y\_{n}\right)\right)$ converges to $\left(x,y\right)\in X×Y$ if and only if $\left(x\_{n}\right)$ converges to $x$ and $\left(y\_{n}\right)$ converges to $y$.

**Question 4:**

1. Define a compact space.
2. Prove that $R$ with Co-finite topology is compact, but $R$ with usual topology is not compact.
3. Prove that any closed set of a compact space is compact.

**Question 5:**

1. Prove that if $f:X\rightarrow R$ is a continuous function from a compact space $X$ into $R$, then $f$ attains its maximal and its minimal.
2. Prove that if $f:X\rightarrow Y$ is a continuous bijection function from a compact space $X$ onto a Hausdorff space $Y$, then $f$ is a homeomorphism.

**Question 6:**

1. Let $X$ be a metrizable space. Prove that $X$ is limit point compact space if and only if $X$ is sequentially compact.
2. If $X$ is not a metrizable space, then prove that the statement in I is not true.

**Bonus:**

Let $d:X×X\rightarrow R$ be a metric on $X$. Prove that for any $x,y,z\in X$

$$\left|d\left(x,y\right)-d(y,z)\right|\leq d\left(x,z\right).$$

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