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| KSU logo tiff.tif |  **King Saud University**  |
|  **College of Sciences** |
|  **Department of Mathematics** |
|  **373 Math** |
|  **Final Exam** |
|  **First semester 1433-1434** |

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**Question 1:**

1. Let $f:X\rightarrow Y$ be a function from a space $X$ into a space $Y$. Prove that $f$ is continuous if and only if for every $A⊆X$, $f(\overbar{A})⊆\overbar{f(A)}$.
2. Let$\left(X\_{1}, τ\_{1}\right)$,…$ ,\left(X\_{n}, τ\_{n}\right)$ be topological spaces, and let $f:Y\rightarrow \prod\_{i=1}^{n}X\_{i}$ be a function from a space $Y$ into a product space $\prod\_{i=1}^{n}X\_{i}$, given by $f\left(y\right)=(f\_{1}\left(y\right),…,f\_{n}\left(y\right))$. Prove that $f$ is continuous if and only if $f\_{1},…,f\_{n}$ are continuous.
3. Let $f:X\rightarrow Y$ and $g:Y\rightarrow Z$ be functions. Prove that if $g∘f$ is open and $f$ is continuous onto function, then $g$ is open.

**Question 2:**

1. Let $\left(X,d\right)$ be a metric space. Prove that the set $\overbar{B}\left(x,ϵ\right)=\left\{y\in X:d(x,y)\leq ϵ\right\}$ is closed, where $x\in X$ and $ϵ>0$. (This set is called the closed ball with center $x$ and radius $ϵ.)$
2. Let $τ$ be the usual topology on $R^{n}$, $n\in N$. Prove that $\left(R^{n},τ\right)$ is a metrizable.

**Question 3:**

1. Prove that in a Hausdorff space any convergent sequence has a unique limit. Give an example to show the converse of the statement does not hold.
2. Let $\left(x\_{n}\right)$ and $\left(y\_{n}\right)$ be sequences in the spaces $X$ and $Y$, respectively. Prove that the sequence $\left(\left(x\_{n},y\_{n}\right)\right)$ converges to $\left(x,y\right)\in X×Y$ if and only if $\left(x\_{n}\right)$ converges to $x$ and $\left(y\_{n}\right)$ converges to $y$.

**Question 4:**

1. Define a compact space.
2. Prove that $R$ with Co-finite topology is compact, but $R$ with usual topology is not compact.
3. Prove that any closed set of a compact space is compact.

**Question 5:**

1. Prove that if $f:X\rightarrow R$ is a continuous function from a compact space $X$ into $R$, then $f$ attains its maximal and its minimal.
2. Prove that if $f:X\rightarrow Y$ is a continuous bijection function from a compact space $X$ onto a Hausdorff space $Y$, then $f$ is a homeomorphism.

**Question 6:**

1. Let $X$ be a metrizable space. Prove that $X$ is limit point compact space if and only if $X$ is sequentially compact.
2. If $X$ is not a metrizable space, then prove that the statement in I is not true.

**Bonus:**

Let $d:X×X\rightarrow R$ be a metric on $X$. Prove that for any $x,y,z\in X$

$$\left|d\left(x,y\right)-d(y,z)\right|\leq d\left(x,z\right).$$

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