<u>373 Math Exercises</u>

Required text book:
Introduction to Topology *by Crump W. Baker*

Chapter 2 :

<u>From Text book:</u>
2.1: 1, 4, 5, 6, 7.
2.2: 1,2, 4, 5, 6, 7, 8, 9

<u>Additional Problems:</u>

1. List all topologies for a set containing three distinct elements.
2. Prove that for a non empty set X, the collection $\tau = \{X, \Phi\} \cup \{U: X - U \ is\ countable\}$ is a topology on X, this topology is called co-countable topology.
3. Is there a set in which discrete and indiscrete topologies coincide on it?
4. Give an example of a nontrivial topology on an infinite set X which has only a finite number of elements.
5. If $\tau_1 \ and\ \tau_2$ are two topologies on X, is $\tau_1 \cap \tau_2$ a topology on X? Is $\tau_1 \cup \tau_2$ a topology on X?
6. Prove that τ is the discrete topology on X iff every point in X is an open set.
7. Let $X = \mathbb{N}$. For each $n \in \mathbb{N}$ define $U_n = \{n, n+1, n+2, \dots \dots .\}$ .Let $\tau = \{X, \phi\} \cup \{U_n : n \in \mathbb{N}\}$. Prove that τ is a topology on X.

<u>From Text book:</u>
2.3:1, 2, 3, 4, 5, 6, 7, 8, 9, 13.
2.4: 1, 2, 3, 4, 5, 6, 7, 10, 13, 14, 15, 16, 17,

Additional Problems:

1. In $(\mathbb{R}, \mathcal{U})$, do rationals form an open set? Closed set? Neither? Both? Justify your answer.
2. Consider $A = \{x: 0 < x < 2\} \cup \{10\}$. Find Cl(A) in $(\mathbb{R}, \mathcal{U})$.
3. Give an example of a collection of open sets whose intersection is not open.
4. Give an example of two sets A and B of In $(\mathbb{R}, \mathcal{U})$such that A and A\B are both open but B is not closed.
5. Give an example of a countable set in In $(\mathbb{R}, \mathcal{U})$ that is not closed.
6. Give an example of a countable set in In $(\mathbb{R}, \mathcal{U})$ that is closed.
7. Prove that $A\ is\ open\ iff\ A \cap Bd(A) = \emptyset$
8. Prove that $A\ is\ closed\ iff\ Bd(A) \subseteq A$.
9. Prove that $Bd(A) = \emptyset\ iff\ A\ is\ both\ open\ and\ closed.$

From Text book:

2.5:1,2,3,4,5,9.10.

Additional Problems:

1.Find a base for the open half-line topology which is different from the topology itself.

2.Let (X, τ) be a topological space and $\mathcal{B}$ a base for τ. Prove that $A \subseteq X$ is dense in X iff each nonempty element of $\mathcal{B}$ contains a point of A.

Definition:

Two collections $\mathcal{B}_1$ and $\mathcal{B}_2$ of subsets of X are equivalent bases iff there exists a topology τ for X such that $\mathcal{B}_1$ and $\mathcal{B}_2$ are both bases for τ.

3.Let (X, τ) be a topological space, $\mathcal{B}_1$a base for τ, and $\mathcal{B}_2$ a collection of subsets of X Prove that $\mathcal{B}_1$ and $\mathcal{B}_2$ are equivalent bases iff

a. For each $U_1 \in \mathcal{B}_1$ and $x \in U_1$, $there$ is a $U_2 \in \mathcal{B}_2$ $such$ $that$ $x \in U_2 \subseteq U_1$ and

b. For each $U_2 \in \mathcal{B}_2$ and $x \in U_2$, $there$ is a $U_1 \in \mathcal{B}_1$ $such$ $that$ $x \in U_1 \subseteq U_2$.

Chapter 3:

From Text book:
3.1: 1,2,3,4,5,6,8,9,13,15,16.
3.2: 1,3,4,6,8,9,10,12,13,14,18.

Additional Problems:

Q1: Let $f: \mathbb{R} \to \mathbb{R}$ be given by $f(x) = \begin{cases} 1 & x \geq 0 \\ -1 & x < 0 \end{cases}$

Detemine whether f is
a) $\tau_{cof} - \mathcal{U}$ continuous.
b) $\mathcal{U} - \tau_{cof}$ continuous.
c) $\mathfrak{F}_1 - \mathcal{U}$ continuous.
d) $\mathcal{U} - \mathfrak{F}_1$ continuous.
e) $C - C$ continuous.

Q2: Repeat the previous question for the function $g\,(x) = x + 1$

From Text book:
3.3: 1,2,3,4,7,8,11,13,14.15.20,21,22.

Additional Problems:
Q2: Let $f: (X,\mathfrak{F}) \to (Y,S)$be a homeomorphism, $A \subseteq X$. Prove the Following:
a) If $a \in int(A)$, then $f(a) \in int\big(f(A)\big)$.
b) If $a \in Bd(A)$, then $f(a) \in Bd\big(f(A)\big)$.

Chapter 4:

From Text book:
4.1: 1,2,3,4,5,6,8,11.
Review Exc.: 3,5

Additional Problems:
Q1: Detemine whether the set $W = \{(x,y) : x > 0, |y| \geq 5\}$ is open in :
(a) $(\mathbb{R},\mathcal{U}) \times (\mathbb{R},\mathcal{H})$
(b) $(\mathbb{R},\mathcal{H}) \times (\mathbb{R},\mathcal{U})$
(c) $(\mathbb{R},C) \times (\mathbb{R},C)$

Chapter 6 :

From Text book:
6.1: 1,2,3,4,5,6,7,8,11,12,13,14,15 ,18,19,20

6.2: 1,2,3,4,5,6,7

Additional Problems:
Q1. Let X be any infinite set with two topologies τ_1 and τ_2 such that (X, τ_2) is a compact space, and $\tau_1 \subset \tau_2$. Show that (X, τ_1) is compact.

Q2.Let $X = \Re,\ \tau = \{U \subseteq \Re; U = \Re \ or\ U = \phi,\ or\ U = (a, \infty),\ a \geq 0\}$ Is $(\Re, \tau)$ a compact space.

Q3. Give an example of a compact space which has a non compact subspace.

Q4. Prove that if (X, τ) is a Hausdorff space then so is every subspace of X.

Q5. Show that Theorem 6.1.21 is not true if X is not compact.

Chapter 8:

From Text book:
8.1: 1, 3, 5, 6, 7, 10, 11.
8.2: 5, 9, 10, 11, 12.
8.3: 2, 3, 4, 5, 8, 9, 14, 15, 16.
8.4: 2(b, c, e, h), 4, 5, 6, 8.

Additional Problems:
Definition: Let $A \neq \varphi$, $B \neq \varphi$ be two subset of the metric space (X, d). Then the distance between A and B is given by $d(A, B) = inf\{d(x, y) : x \in A, y \in B\}$.
If $A = a$, we write $d(a, B)$ for $d(A, B)$.

Q1. Let (X, d) be a metric space and A a nonempty subset of X. If $x, y \in X$, prove that $d(x, A) \leq d(x, y) + d(y, A)$.
Q2. Give an example to show that for an open ball $B_r(y)$ in a metric space (X,d), it is not true that $Bd(B_r(y)) = \{x \in X : d(x,y) = r\}$
Q3. Let (X,d) be a metric space and define $e(x,y) = d(x,y)/(1 + d(x,y))$ Prove that e is a bounded metric for X.