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STAT 340

THEORY OF STATISTICS

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Chapter 1: ***Introduction***

1.1: ***Suppose that 4 out of 12 buildings in a certain city violate the building code. A building engineer randomly inspects a sample of 3 new buildings in the city*.**

$$\text{given that } p = \frac{4}{12} = \frac{1}{3} \quad \Rightarrow q = \frac{2}{3} \text{ and } n = 3$$

(a) Find the probability distribution function of the random variable X representing the number of buildings that violate the building code in the sample.

$$\boldsymbol{P(X = x)} = \binom{n}{x} p^x q^{n-x},\ x = 0,1,\dots,n$$

$$P(X = x) = \binom{3}{x}\left(\frac{1}{3}\right)^x \left(\frac{2}{3}\right)^{3-x},\ x = 0,1,2,3$$

(b) Find the probability that

(i) ***none of the buildings in the sample violating the building code***

$$\boldsymbol{P(X = 0)} = \binom{3}{0}\left(\frac{1}{3}\right)^0 \left(\frac{2}{3}\right)^{3-0} = 0.2963$$

(ii) ***one building in the sample violating the building code*.**

$$\boldsymbol{P(X = 1)} = \binom{3}{1}\left(\frac{1}{3}\right)^1 \left(\frac{2}{3}\right)^{3-1} = 0.4444$$

(iii) ***at lease one building in the sample violating the building code.***

$$\boldsymbol{P(X \geq 1)} = 1 - P(X < 1) = 1 - P(X = 0) = 1 - 0.2963 = 0.7037$$

(c) Find the expected number of buildings in the sample that violate the building code.

$$\mu = np = 3\left(\frac{1}{3}\right) = 1$$

(d) Find Var(X).

$$\sigma^2 = npq = 3\left(\frac{1}{3}\right)\left(\frac{2}{3}\right) = \frac{2}{3}$$

1.2: ***On average, a certain intersection results in 3 traffic accidents per day. Assuming Poisson distribution***

λ: $average\ number\ of\ traffic\ accident\ per\ day$

λ: $3\ per\ day$

X: $number\ of\ traffic\ accident\ per\ day$

$$P(X = x) = \frac{e^{-\lambda}\lambda^x}{x!} = \frac{e^{-3}3^x}{x!}, \qquad x = 0,1,2,\ldots$$

(i) ***what is the probability that at this intersection***

(a) no accidents will occur in a given day?

$$P(X = 0) = \frac{e^{-3}3^0}{0!} = 0.049787$$

(b) More than 3 accidents will occur in a given day?

$$P(X > 3) = 1 - P(X \le 3) = 1 - \left(\sum_{x=0}^{3} \frac{e^{-3}3^x}{x!}\right) = 0.3528$$

(c) Exactly 5 accidents will occur in a period of two days?

$\lambda_1 = average\ number\ of\ traffic\ accidents\ per\ 2\ days$

$\lambda_1 = \lambda t = 3(2) = 6\ \ per\ two\ days$

$$P(Y = y) = \frac{e^{-\lambda_1}\lambda_1{}^y}{y!} = \frac{e^{-6}6^y}{y!}, \ \ y = 0,1,2,\ldots$$

$$P(Y = 5) = \frac{e^{-6}6^5}{5!} = 0.1606$$

(ii) ***what is the average number of traffic accidents in a period of 4 days?***

$\lambda_2 = average\ number\ of\ traffic\ accidents\ per\ 4\ days$

$\lambda_2 = \lambda t = 3(4) = 12\ per\ 4\ days$

1.3: ***If the random variable X has a uniform distribution on the interval (0,10), then***

Given that $X \sim Uniform(0,10) \Rightarrow f_X(x) = \frac{1}{b-a} = \frac{1}{10-0} = \frac{1}{10}, \ 0 \le x \le 10$

(a) $P(X < 6) = \int_0^6 \frac{1}{10} dx = \frac{3}{5}$

(b) ***The mean of X is*** $\mu = \frac{a+b}{2} = \frac{10}{2} = 5$

(c) ***The variance X is*** $\sigma^2 = \frac{(b-a)^2}{12} = 8.333$

1.4: ***Suppose that Z is distributed according to the standard normal distribution.***

(a) the area under the curve to the left of 1.43 is:

$P(Z < 1.43) = 0.9236$

(b) the area under the curve to the right of 0.89 is:

$P(Z > 0.89) = P(Z < -0.89) = 0.1867$

or $P(Z > 0.89) = 1 - P(Z < 0.89) = 1 - 0.8133 = 0.1867$

(c) the area under the curve between 2.16 and 0.65 is:

$P(0.65 < Z < 2.16) = P(Z < 2.16) - P(Z < 0.65) = 0.9846 - 0.7422 = 0.2424$

(d) the value of k such that P(0.93< Z < k) = 0.0427 is:

$P(0.93 < Z < k) = 0.0427$

$\Leftrightarrow \ P(Z < k) - P(Z < 0.93) = 0.0427$

$\Leftrightarrow \ P(Z < k) - 0.8238 = 0.0427$

$\Leftrightarrow \ P(Z < k) = 0.8665$

$\Leftrightarrow \ k = 1.11$

1.5: ***The finished inside diameter of a piston ring is normally distributed with a mean of 12 centimeters and a standard deviation of 0.03 centimeter.*** $X \sim Normal(12, (0.03)^2)$

عادة ما يتم توزيع القطر الداخلي النهائي لحلقة المكبس بمتوسط 12 سم وانحراف معياري يبلغ 0.03 سم.

Find:

(a) the proportion of rings that will have inside diameter <u>less than 12.05</u> centimeters.

$$\boldsymbol{P(X < 12.05)} = P\left(Z < \frac{12.05-12}{0.03}\right) = P(z < 1.67) = 0.9525$$

(b) the proportion of rings that will have inside diameter <u>exceeding 11.97</u> centimeters.

$$\boldsymbol{P(X > 11.97)} = P\left(Z > \frac{11.97-12}{0.03}\right) = P(Z > -1) = P(Z < 1) = 0.8413$$

(c) the probability that a piston ring will have an inside diameter between 11.95 and 12.05 centimeters.

$$P(11.95 < X < 12.05) = P\left(\frac{11.95-12}{0.03} < Z < \frac{12.05-12}{0.03}\right)$$

$$= P(z < -1.67) - P(z < 1.67)$$

$$= 0.9525 - 0.0475 = 0.905$$

1.6: ***Let X be $N(\mu,\sigma^2)$ so that $P(X<89)=0.90$ and $P(X<94)=0.95$. find μ and σ^2.***

$X \sim Normal(\mu, \sigma^2)$

$P(X < 89) = 0.9$

$P\left(Z < \frac{89-\mu}{\sigma}\right)=0.9$

$\frac{89-\mu}{\sigma} = 1.28$

$89 - \mu = 1.28\,\sigma$

$\mu = 89 - 1.28\,\sigma$ ...(1)

$P(X < 94) = 0.95$

$P\left(Z < \frac{94-\mu}{\sigma}\right)=0.95$

$\frac{94-\mu}{\sigma} = 1.645$

$94 - \mu = 1.645\,\sigma$

$\mu = 94 - 1.645\,\sigma$ ...(2)

$P(Z < 1.64) = 0.9495$

$P(Z < 1.65) = 0.9505$

$\frac{1.65 + 1.64}{2} = 1.645$

Then, $89 - 1.28\,\sigma = 94 - 1.645\,\sigma$

$(1.645-1.28)\,\sigma=94-89$

$\Rightarrow \sigma = 13.6986,$

$\Rightarrow \sigma^2=187.65$

We substitute in (1) or (2) by $\sigma = 13.6986$ *we get*

$\Rightarrow \mu = 71.46575$

1.7: ***Assume the length (in minutes) of a particular type of a telephone conversation is a random variable with a probability density function of the form:***

$$f(x) = 0.2e^{-0.2x}, x > 0 \Rightarrow X \sim exp\left(\frac{1}{\theta} = \frac{1}{0.2}\right)$$

Calculate:

(a) $P(3 < x < 10) = \int_3^{10} 0.2e^{-0.2x}dx = 0.4135$

(b)The cdf of X. $F(x) = 1 - e^{-\theta x} = 1 - e^{-0.2x}$

(c) The mean and the variance of X.

$$\mu = \frac{1}{\theta} = \frac{1}{0.2} = 5 \qquad \sigma^2 = \frac{1}{\theta^2} = \frac{1}{0.2^2} = 25$$

1.8: ***Find the moment-generating function of X, if you know that***

$$f(x) = 2e^{-2x},\ x > 0 \Rightarrow X \sim exp\left(\frac{1}{\theta} = \frac{1}{2}\right)$$

$M_X(t) = \frac{\theta}{\theta - t} = \frac{2}{2-t}$ *or* $M_X(t) = \left(1 - \frac{t}{\theta}\right)^{-1} = \left(1 - \frac{t}{2}\right)^{-1}$ *where,* $t < \theta \Rightarrow t < 2$

1.9: ***For a chi-squared distribution, find***

${\chi^2}_{0.025,15} = 27.49$

${\chi^2}_{0.01,7} = 18.48$

${\chi^2}_{0.99,22} = 9.54$

1.10: ***If*** $(1-2t)^{-6}$***, t<12, is the MGF of the random variable X, find P(X<5.23).***

given that $M_X(t) = (1-2t)^{-6}$,

We know that if $X \sim {\chi^2}_v \Rightarrow M_X(t) = (1-2t)^{-\frac{v}{2}}$,

$\Rightarrow \frac{v}{2} = 6 \Rightarrow v = 12$

$P(X < 5.23) = P\left({\chi^2}_{12} < 5.23\right) = 1 - P\left({\chi^2}_{12} > 5.23\right) = 1 - 0.950 = 0.05$

<u>1.11:</u> ***Find:***

(a) $\boldsymbol{t_{0.95,28}} = 1.701$

(b) $\boldsymbol{t_{0.005,16}} = -t_{0.995,16} = -2.921$

(c) $-\boldsymbol{t_{0.01,4}} = -\left(-t_{0.01,4}\right) = t_{0.99,4} = 3.747$

(d) $\boldsymbol{P(T_{24} > 1.318)} = 1 - P(T_{24} < 1.318) = 1 - 0.9 = 0.1$

(e) $\boldsymbol{P(-1.356 < T_{12} < 2.179)} = P(T_{12} < 2.179) - P(T_{12} < -1.356)$

$$= P(T_{12} < 2.179) - P(T_{12} > 1.356)$$

$$= 0.975 - 0.1$$

$$= 0.875$$

<u>1.12:</u> ***If*** $f(x) = \theta x^{\theta-1}$, $0 < x < 1$, ***find the distribution of Y=−lnX.***

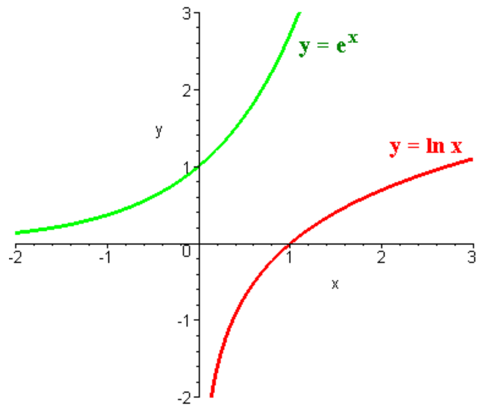
Since $0 < x < 1$

$\Rightarrow ln0 < lnx < ln1$

$\Rightarrow -\infty < lnx < 0$

$\Rightarrow \infty > -lnx > 0$

$\Rightarrow \boxed{0 < y < \infty}$



- *By using one to one transformation methods:*

$y = -lnx$

$\Rightarrow -y = lnx$

$\Rightarrow x = e^{-y}$

$f_y(y) = f_x[w(y)]|J|$

$f_x[w(y)]$ $= f_x[e^{-y}]$ $= \theta(e^{-y})^{\theta-1}$	$\lvert J\rvert = \left\lvert \frac{d}{dy} x = e^{-y} \right\rvert$ $= \lvert -e^{-y}\rvert$ $= e^{-y}$

$f_y(y) = f_x[w(y)]|J|$

$f_y(y) = \theta(e^{-y})^{\theta-1} e^{-y}$

$f_y(y) = \theta e^{-y(\theta-1)} e^{-y}$

$f_y(y) = \theta e^{-\theta y} e^{y} e^{-y}$

$\boxed{f_y(y) = \theta e^{-\theta y} \; ; \; 0 < y < \infty}$ $\quad Y \sim exp\left(\frac{1}{\theta}\right)$

- *By using CDF methods:*

$F_X(x) = P(X < x) = \int_0^x f(x)dx = \int_0^x \theta x^{\theta-1} dx$

$= \theta \int_0^x x^{\theta-1} dx$

$= \theta \left[\frac{x^\theta}{\theta}\right]_0^x = x^\theta$

Then, $F_Y(y) = P(Y < y) = P(-lnX < y)$

$= P(lnX > -y)$

$= P(X > e^{-y})$

$= 1 - P(X < e^{-y})$

$= 1 - F_X(e^{-y})$

$= 1 - (e^{-y})^\theta$

$= 1 - e^{-\theta y} = F_Y(y)$

We know that $f_Y(y) = \frac{d}{dy} F_Y(y)$

$= \frac{d}{dy}\left(1 - e^{-\theta y}\right)$

$= \theta e^{-\theta y}$

$$\boxed{f_y(y) = \theta e^{-\theta y} \quad ; \quad 0 < y < \infty} \qquad Y \sim exp\left(\frac{1}{\theta}\right)$$

1.13: **If $f(x) = 1,\ 0 < x < 1$. Find the pdf of $Y=\sqrt{x}$.**

Since $0 < x < 1$

$\Rightarrow 0 < \sqrt{x} < 1$

$\Rightarrow \boxed{0 < y < 1}$

- *By using one to one transformation methods:*

$y = \sqrt{x} \quad \Rightarrow x = y^2$

$$f_y(y) = f_x[w(y)]\ |J|$$

$f_x[w(y)]$ $= f_x[\,y^2\,]$ $= 1$	$\lvert J\rvert = \left\lvert \frac{d}{dy} x = y^2 \right\rvert$ $= \lvert 2y\rvert$ $= 2y$

$$f_y(y) = f_x[w(y)]\ |J|$$

$$f_y(y) = 2y$$

$$\boxed{f_y(y) = 2y \quad ; \quad 0 < y < 1}$$

- *By using CDF methods:*

$F_X(x) = \int_0^x f(x)dx = \int_0^x dx = x$

Then, $F_Y(y) = P(Y < y) = P(\sqrt{X} < y)$

$= P(X < y^2)$

$= F_X(y^2) = y^2$

We know that $f_Y(y) = \frac{d}{dy} F_Y(y)$

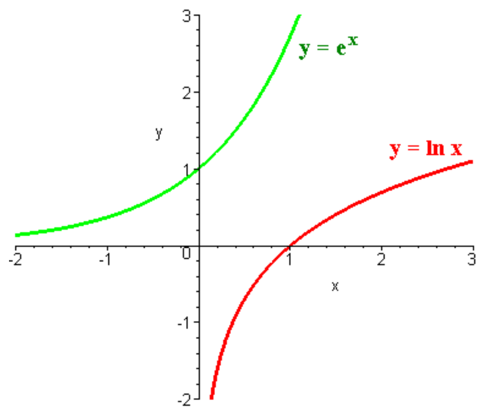
$= \frac{d}{dy} y^2 = 2y$

$$\boxed{f_Y(y) = 2y \quad ; \quad 0 < y < 1}$$

<u>1.14:</u> ***If $X \sim U(0,1)$, find the pdf of $Y = -2\ln X$.***
Name the distribution and its parameter values.

$f(x) = 1, \; 0 < x < 1$

$$\text{Since } 0 < x < 1$$
$$\Rightarrow \ln 0 < \ln x < \ln 1$$
$$\Rightarrow -\infty < \ln x < 0$$
$$\Rightarrow \infty > -2\ln x > 0$$
$$\Rightarrow \boxed{0 < y < \infty}$$



- *By using one to one transformation methods:*

$$y = -2\ln x$$
$$\Rightarrow -\frac{y}{2} = \ln x$$
$$\Rightarrow x = e^{-\frac{y}{2}}$$

$$f_y(y) = f_x[w(y)]\,|J|$$

$f_x[w(y)]$ $= f_x\left[e^{-\frac{y}{2}}\right]$ $= 1$	$\lvert J \rvert = \left\lvert \frac{d}{dy} x = e^{-\frac{y}{2}} \right\rvert$ $= \left\lvert -\frac{1}{2} e^{-\frac{y}{2}} \right\rvert$ $= \frac{1}{2} e^{-\frac{y}{2}}$

$$f_y(y) = f_x[w(y)]\,|J|$$
$$f_y(y) = \frac{1}{2} e^{-\frac{y}{2}}$$
$$\boxed{f_y(y) = \frac{1}{2} e^{-\frac{y}{2}} \;\; ; \;\; 0 < y < \infty} \qquad Y \sim exp\,(2)$$

- *By using CDF methods:*

$F_X(x) = \int_0^x f(x)dx = \int_0^x dx = x$

Then, $F_Y(y) = P(Y < y) = P(-2lnX < y)$

$$= P\left(lnX > -\frac{y}{2}\right)$$

$$= P\left(X > e^{-\frac{y}{2}}\right)$$

$$= 1 - P\left(X < e^{-\frac{y}{2}}\right)$$

$$= 1 - F_X\left(e^{-\frac{y}{2}}\right)$$

$$= 1 - e^{-\frac{y}{2}}$$

We know that $f_Y(y) = \frac{d}{dy}F_Y(y)$

$$= \frac{d}{dy}\left(1 - e^{-\frac{y}{2}}\right)$$

$$= \frac{1}{2}e^{-\frac{y}{2}}$$

$$\boxed{f_y(y) = \frac{1}{2}e^{-\frac{y}{2}} \quad ; \quad 0 < y < \infty} \qquad Y \sim exp\,(2)$$

1.15: **Suppose independent random variables X and Y are such that** $M_{X+Y}(t) = \frac{e^{2t}-1}{2t-t^2}$.

If and $f_X(x) = 2e^{-2x},\ x > 0$**, what is the distribution of Y.**

Given that $f_X(x) = 2e^{-2x},\ x > 0$

$$x \sim exp\left(\frac{1}{2}\right) \Longrightarrow M_X(t) = \frac{\theta}{\theta - t} = \frac{2}{2-t}$$

As X and Y independent $\Rightarrow M_{X+Y}(t) = M_X(t)M_Y(t)$

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$M_Y(t) = \frac{M_{X+Y}(t)}{M_X(t)} = \frac{M_X(t)M_Y(t)}{M_X(t)}$ $= \frac{e^{2t}-1}{2t-t^2} / \frac{2}{2-t}$ $= \frac{e^{2t}-1}{t(2-t)} \frac{2-t}{2}$ $= \frac{e^{2t}-1}{2t}$ $= \frac{e^{2t}-e^{0t}}{t(2-0)}$ *The MGF of uniform* $= \frac{e^{bt}-e^{at}}{t(b-a)}$	$M_{X+Y}(t) = \frac{e^{2t}-1}{2t-t^2} = \frac{e^{2t}-1}{t(2-t)}$ $= \frac{e^{2t}-1}{t} \frac{1}{(2-t)}$ $= \frac{e^{2t}-1}{2t} \frac{2}{(2-t)}$ $= \frac{e^{2t}-1}{2t} \boxed{\frac{2}{(2-t)}}$ $= \frac{e^{2t}-1}{2t} \boxed{M_X(t)}$ $= \frac{e^{2t}-e^{0t}}{t(2-0)}$ *The MGF of uniform* $= \frac{e^{bt}-e^{at}}{t(b-a)}$

then $Y \sim uniform\ (a = 0, b = 2)$

1.16: **If** $X_1 \sim \chi^2_n$ **and** $X_2 \sim \chi^2_m$ **are independent random variables.**

Find the distribution of $Y = X_1 + X_2$

We know that if $X \sim \chi^2_v \Longrightarrow M_X(t) = (1-2t)^{-\frac{v}{2}}$

$$M_Y(t) = M_{X_1+X_2}(t) = M_{X_1}(t)M_{X_2}(t)$$

$$= (1-2t)^{-\frac{n}{2}}(1-2t)^{-\frac{m}{2}}$$

$$= (1-2t)^{-\frac{n+m}{2}}$$

Which is the MGF of $y \sim \chi^2_{n+m}$

Chapter 2: Sampling Distribution

2.1: **If e^{3t+4t^2} is the MGF of the random variable $\bar{X}$ with sample size 6, find $P(-2 < \bar{X} < 6)$**

Given that, $M_{\bar{X}}(t) = e^{3t+4t^2} \Longrightarrow \bar{X} \sim N(\mu_{\bar{X}}, \sigma^2_{\bar{X}})$

know that if $X \sim N(\mu_X, \sigma^2_X) \Longrightarrow M_X(t) = e^{\mu t + \frac{1}{2}\sigma^2 t^2}$

$$\mu t + \frac{1}{2}\sigma^2 t^2 = 3t + 4t^2$$

$\mu_{\bar{X}} = 3$	$\frac{1}{2}\sigma^2_{\bar{X}} = 4 \Longrightarrow \sigma^2_{\bar{X}} = 8$

$\Longrightarrow \bar{X} \sim N(3,8)$

$$P(-2 < \bar{X} < 6) = P\left(\frac{-2-3}{\sqrt{8}} < Z < \frac{6-3}{\sqrt{8}}\right)$$

note that $\sigma^2_{\bar{X}} = \frac{\sigma^2_X}{n}$

$$= P(-1.77 < Z < 1.06)$$

$$= P(Z < 1.06) - P(Z < -1.77)$$

$$= 0.8554 - 0.0384$$

$$= 0.8170$$

2.2: **Let X be the mean of a random sample of size 5 from a normal distribution with $\mu = 0$ and $\sigma^2 = 125$. Find *c* so that $P(\bar{X} < c) = 0.975$**

given that $X \sim N(0,125)$, $n = 5$ and $P(\bar{X} < c) = 0.975$

$$X \sim N(\mu, \sigma^2) \Longrightarrow \bar{X} \sim N(\mu, \frac{\sigma^2}{n})$$

Since $X \sim N(0,125) \Longrightarrow \bar{X} \sim N(0, \frac{125}{5}) \Longrightarrow \bar{X} \sim N(0,25)$

$$P(\bar{X} < c) = 0.975 \quad \Rightarrow P\left(Z < \frac{c-0}{\sqrt{25}}\right) = 0.975$$

$$\Rightarrow \frac{c}{\sqrt{25}} = 1.96 \quad \Rightarrow \boxed{c = 9.8}$$

2.3: **Determine the mean and variance of the mean $\overline{X}$ of a random sample of size 9 from a distribution having pdf** $f(x) = \begin{cases} 4x^3, & 0 < x < 1 \\ 0 & \text{, else where} \end{cases}$

We want to find $\mu_{\overline{X}}$ and $\sigma^2_{\overline{X}}$, we have to find first μ_X and σ^2_X

$E(X) = \int_0^1 x\, f(x) dx$ $= \int_0^1 x\, 4x^3 dx$ $= 4\int_0^1 x^4\, dx$ $= 4\left[\frac{x^5}{5}\right]_0^1 = \frac{4}{5}$	$E(X^2) = \int_0^1 x^2\, f(x)$ $= \int_0^1 x^2\, 4x^3 dx$ $= 4\int_0^1 x^5\, dx$ $= 4\left[\frac{x^6}{6}\right]_0^1 = \frac{4}{6}$

$$E(X) = \mu_X = \mu_{\overline{X}} = \frac{4}{5}$$

$$\sigma^2_X = E(X^2) - \left(E(X)\right)^2$$

$$= \frac{4}{6} - \left(\frac{4}{5}\right)^2 = \frac{2}{75}$$

$$\Longrightarrow \sigma^2_{\overline{X}} = \frac{\sigma^2_X}{n} = \frac{\frac{2}{75}}{9} = \frac{2}{675}$$

$$\boxed{\mu_{\overline{X}} = \frac{4}{5} \qquad \sigma^2_{\overline{X}} = \frac{2}{675}}$$

2.4: **Let $Z_1, Z_2, \dots, Z_{16}$ be a random sample of size 16 from the standard normal distribution (0, 1). Let $X_1, X_2, \dots, X_{64}$ be a random sample of size 64 from the normal distribution (μ, 1). The two samples are independent.**

(a) $P(Z_1 < 2) = 0.9772$

(b) $P\left(\sum_{i=1}^{16} Z_i > 2\right)$

$\sum_{i=1}^{n} a_i Z_i \sim N(\sum a_i\mu_i \ , \sum a_i^2\sigma_i^2)$
$\sum_{i=1}^{16} \ Z_i \sim N(\sum \mu_i \ , \sum \sigma_i^2)$
$\sum_{i=1}^{16} \ Z_i \sim N(16\mu, 16\sigma^2)$

$\boxed{if\ X_1, X_2, \dots, X_n\ iid \sim N(\mu, \sigma^2) \quad \sum a_i X_i \sim N(\sum a_i\mu_i \ , \sum a_i^2\sigma_i^2\ )}$ *p. 30*

$$\sum_{i=1}^{16} Z_i \sim N(16(0), 16(1))$$

$$\sum_{i=1}^{16} Z_i \sim N(0,16)$$

$$P(\sum_{i=1}^{16} Z_i > 2) = P\left(Z > \frac{2-0}{\sqrt{16}}\right) = P(Z > 0.5)$$

$$= P(Z < -0.5) = 0.3085$$

$P(Z > a) = 1 - P(Z < a)$
or $P(Z > a) = \ \ P(Z < -a)$

(c) $\mathbf{P\left(\sum_{i=1}^{16} Z_i^2 > 6.91\right)}$

We know that if $Z_i \sim N(0,1), i = 1,2,..,n$

$$\Rightarrow Z_i^2 \sim \chi_1^2$$

$$\Rightarrow \sum_{i=1}^{n} Z_i^2 \sim \chi_n^2$$

Then, $P(\sum_{i=1}^{16} Z_i^2 > 6.91)$

$$= P(\chi_{16}^2 > 6.91) = 0.975$$

(d) **Find c such that** $\mathbf{P(S^2 > c) = 0.05}$ $\boxed{\frac{(n-1)S^2}{\sigma^2} \sim \chi_{n-1}^2}$ p. 36

Since $Z_i \sim N(0,1), \; i = 1,2,..,16$

Then, $P(S^2 > c) = 0.05$

$$\Rightarrow P\left(\frac{(n-1)}{\sigma^2} S^2 > \frac{(n-1)}{\sigma^2} c\right) = 0.05$$

$$\Rightarrow P\left(\frac{(16-1)}{1} S^2 > \frac{(16-1)}{1} c\right) = 0.05$$

$$\Rightarrow P(\chi_{15}^2 > 15c) = 0.05$$ from χ_n^2 table

$$\Rightarrow 15c = 25 \Rightarrow c = \frac{25}{15} \Rightarrow \boxed{c = \frac{5}{3}}$$

(e) **What is the distribution** of $\mathbf{Y = \sum_{i=1}^{16} Z_i^2 + \sum_{i=1}^{64} (X_i - \mu)^2}$

$\boxed{\text{And we know that if } X \sim \chi_n^2 \text{ and } Y \sim \chi_m^2 \implies X + Y \sim \chi_{n+m}^2}$ p. 37

Then, $\mathbf{Y} = \sum_{i=1}^{16} Z_i^2 + \sum_{i=1}^{64} (X_i - \mu)^2$

We know from (c) that	Since $X_i \sim N(\mu, 1)$
$\sum_{i=1}^{16} Z_i^2 \sim \chi_{16}^2$	$\Rightarrow \frac{X_i - \mu}{1} \sim N(0,1)$ $\Rightarrow \left(\frac{X_i - \mu}{1}\right)^2 \sim \chi_1^2$ $\Rightarrow \sum_{i=1}^{64} (\frac{X_i - \mu}{1})^2 \sim \chi_{64}^2$

$$\mathbf{Y} = \sum_{i=1}^{16} Z_i^2 + \sum_{i=1}^{64} (X_i - \mu)^2$$

$$\mathbf{Y} = \boldsymbol{\chi_{16}^2 + \chi_{64}^2} \sim \boxed{\boldsymbol{\chi_{80}^2}}$$

(f) $\mathbf{E(Y) = 80}$

(g) $\mathbf{Var(Y) = 2 \times 80 = 160}$

(h) $\mathbf{P(Y > 105)} = P(\chi^2_{80} > 105) = 0.025$

(i) **Find c such that** $c\frac{\sum_{i=1}^{16} Z_i^2}{Y} \sim F_{16,80}$

we know that $\sum_{i=1}^{16} Z_i^2 \sim \chi^2_{16}$

$Y \sim \chi^2_{80}$

$\boxed{\frac{\chi^2_{v_1}/v_1}{\chi^2_{v_2}/v_2} \sim F_{v_1,v_2}}$ p. 40

$$\frac{\sum_{i=1}^{16} Z_i^2}{Y} = \frac{\chi^2_{16}}{\chi^2_{80}}$$

$$\Rightarrow \frac{\frac{1}{16}}{\frac{1}{80}}\frac{\chi^2_{16}}{\chi^2_{80}} = \frac{80}{16}\frac{\chi^2_{16}}{\chi^2_{80}} \sim F_{16,80} \Longrightarrow c = \frac{80}{16} \Longrightarrow \boxed{c = 5}$$

(j) **Let** $Q \sim \chi^2_{60}$ **find c such that** $P\left(\frac{Z_1}{\sqrt{Q}} < c\right) = 0.95$

We know that $Z_1 \sim N(0,1)$ and given that $Q \sim \chi^2_{60}$ then

$\boxed{\frac{Z_1}{\sqrt{Q/v}} \sim t_v}$ p. 38

$$P\left(\frac{Z_1}{\sqrt{Q}} < c\right) = 0.95 \Rightarrow P\left(\frac{Z_1}{\frac{1}{\sqrt{60}}\sqrt{Q}} < \frac{c}{\frac{1}{\sqrt{60}}}\right) = 0.95$$

$$\Rightarrow P\left(\frac{Z_1}{\sqrt{Q/60}} < \frac{c}{1/\sqrt{60}}\right) = 0.95$$

$$\Rightarrow P\left(t_{60} < c\sqrt{60}\right) = 0.95$$

$$\Rightarrow c\sqrt{60} = 1.671 \Rightarrow \boxed{c = 0.2157}$$

(k) **Find c such that** $P\left(F_{60,20} > c\right) = 0.99$

$$c = F_{0.99,60,20} = \frac{1}{F_{0.01,20,60}}$$

$\boxed{F_{1-\alpha}(v_1, v_2) = \frac{1}{F_{\alpha}(v_2,v_1)}}$ p. 21

$$= \frac{1}{2.20} = 0.4545$$

2.5: **Let $X \sim N(5,10)$ find $P(0.04 < (X-5)^2 < 38.4)$**

Since $X \sim N(5,10) \quad \Rightarrow \frac{X-5}{\sqrt{10}} \sim N(0,1) \Rightarrow \left(\frac{X-5}{\sqrt{10}}\right)^2 \sim \chi_1^2$

Then, $P(0.04 < (X-5)^2 < 38.4) = P\left(\frac{1}{(\sqrt{10})^2}0.04 < \frac{1}{(\sqrt{10})^2}(X-5)^2 < \frac{1}{(\sqrt{10})^2}38.4\right)$

$$= P\left(\frac{0.04}{(\sqrt{10})^2} < \left(\frac{X-5}{\sqrt{10}}\right)^2 < \frac{38.4}{(\sqrt{10})^2}\right)$$

$$= P(0.004 < \chi_1^2 < 3.84)$$

$$= P(\chi_1^2 < 3.84) - P(\chi_1^2 < 0.004)$$

$$= 1 - P(\chi_1^2 > 3.84) - [1 - P(\chi_1^2 > 0.004)]$$

$$= P(\chi_1^2 > 0.004) - P(\chi_1^2 > 3.84)$$

$$= 0.95 - 0.05 = 0.90$$

2.6: **Let S^2 be the variance of a random sample of size 6 from the normal distribution $(\mu, 12)$.**

Given that $X \sim N(\mu, 12) \quad n = 6$

(a) $\mathbf{E(S^2)} = \sigma^2 = 12$

$\mathbf{Var(S^2)} = \frac{2\sigma^4}{n-1} = \frac{2(\sigma^2)^2}{6-1} = \frac{2(12)^2}{5} = 57.6$ $\quad \boxed{Var(S^2) = \frac{2\sigma^4}{n-1}}$ p. 41

(b) Distribution of S^2

We know that $X \sim N(\mu, 12) \quad \Rightarrow \frac{(n-1)S^2}{\sigma^2} \sim \chi_{n-1}^2$ $\quad \boxed{\frac{(n-1)S^2}{\sigma^2} \sim \chi_{n-1}^2}$ p. 36

$$\Rightarrow \frac{(6-1)S^2}{12} \sim \chi_{n-1}^2 \quad \Rightarrow \frac{5S^2}{12} \sim \chi_5^2$$

(c) $\mathbf{P(2.30 < S^2 < 22.2)}$

We have from (b) $\frac{5S^2}{12} \sim \chi_5^2$ $\qquad P\left(\frac{5}{12}2.30 < \frac{5}{12}S^2 < \frac{5}{12}22.2\right)$

$$= P(0.96 < \chi_5^2 < 9.25)$$

$$= P(\chi_5^2 > 0.96) - P(\chi_5^2 > 9.25)$$

$$= 0.975 - 0.1 = 0.875$$

2.7: **Let X_1, X_2 and X_3 be iid random variable, each with pdf $f(x) = e^{-x}$; $0 < x < \infty$ and let $Y_1 < Y_2 < Y_3$ be the order statistics of the random variables. Find:**

a. the distribution of $Y_1 = \min(X_1, X_2, X_3)$

Since the pdf is $f_X(x) = e^{-x} \Rightarrow X \sim \exp(1) \Rightarrow F_X(x) = 1 - e^{-x}$

$$\boxed{f_{Y_1}(y_1) = n\, f_X(y_1)[1 - F_X(y_1)]^{n-1};\ \ 0 < y_1 < \infty}\ \text{p. 42}$$

$$= 3e^{-y_1}[1 - (1 - e^{-y_1})]^{3-1}$$

$$= 3e^{-y_1}e^{-2y_1}$$

$$= 3e^{-3y_1}$$

$$\boxed{f_{Y_1}(y_1) = 3e^{-3y_1}\ ;\ 0 < y_1 < \infty} \qquad Y_1 \sim \exp\left(\frac{1}{3}\right)$$

b. $\mathbf{P(3 \le Y_1) =}$ $\mathbf{P(Y_1 \ge 3)} = 1 - P(Y_1 < 3)$

$$= \left[1 - \left(1 - e^{-3(3)}\right)\right] = e^{-9} = 0.00012$$

c. The joint pdf of Y_2 and Y_3

$$\boxed{f_{r,k}(y_r, y_k) = \frac{n!}{(r-1)!\,(k-r-1)!\,(n-k)!}[F_X(y_r)]^{r-1}[F_X(y_k) - F_X(y_r)]^{k-r-1}[1 - F_X(y_k)]^{n-k} f_X(y_k) f_X(y_r)}\ \text{p. 43}$$

We have here $r = 2, k = 3$ and $n = 3$

$$\boxed{\begin{array}{ll} f(y_2) = e^{-y_2}, & F(y_2) = 1 - e^{-y_2} \\ f(y_3) = e^{-y_3}, & F(y_3) = 1 - e^{-y_3} \end{array}}$$

$$f_{2,3}(y_2, y_3) = \frac{3!}{(2-1)!(3-2-1)!(3-3)!}[F(y_2)]^{2-1}[F(y_3) - F(y_2)]^{3-2-1}[1 - F(y_3)]^{3-3} f(y_3) f(y_2)$$

$$f_{2,3}(y_2, y_3) = \frac{3!}{(2-1)!\underbrace{(3-2-1)!}_{1}\underbrace{(3-3)!}_{1}}[F(y_2)]^1 \underbrace{[F(y_3) - F(y_2)]^{3-2-1}}_{1}\underbrace{[1 - F(y_3)]^{3-3}}_{1} f(y_3) f(y_2)$$

$$= 6[1 - e^{-y_2}]e^{-y_3}e^{-y_2}$$

$$= 6[1 - e^{-y_2}]e^{-(y_2 + y_3)}\ ;\ \ 0 < y_2 < y_3 < \infty$$

2.8: **Let $Y_1 < Y_2 < \cdots < Y_n$ be the order statistics from a Weibull distribution. Find the distribution function and pdf of Y_1.**

If $X \sim \text{Weibull}\left(\alpha, \frac{1}{\beta}\right)$

$$\Rightarrow f_X(x) = \alpha\beta x^{\beta-1} e^{-\alpha x^{\beta}}$$

$$\Rightarrow F_X(x) = 1 - e^{-\alpha x^{\beta}}, \quad x \geq 0,\ \alpha > 0,\ \beta > 0$$

$$\boxed{f_{Y_1}(y_1) = n\, f_X(y_1)[1 - F_X(y_1)]^{n-1};\ \ 0 < y_1 < \infty}\ \text{p. 42}$$

$$= n\, \alpha\beta {y_1}^{\beta-1} e^{-\alpha {y_1}^{\beta}} \left[1 - \left(1 - e^{-\alpha {y_1}^{\beta}}\right)\right]^{n-1}$$

$$= n\alpha\beta {y_1}^{\beta-1} e^{-\alpha {y_1}^{\beta}} \left[e^{-\alpha {y_1}^{\beta}}\right]^{n-1}$$

$$= n\alpha\beta {y_1}^{\beta-1} \left[e^{-\alpha {y_1}^{\beta}}\right]^{n}$$

$$= n\alpha\beta {y_1}^{\beta-1} e^{-n\alpha {y_1}^{\beta}}$$

$$\Rightarrow Y_1 \sim \text{Weibull}\left(n\alpha, \frac{1}{\beta}\right)$$

Then, $F_{Y_1}(y_1) = 1 - e^{-n\alpha x^{\beta}}$.

Chapter 3: Point Estimation

3.1: **Suppose $X_1, X_2, \dots, X_n$ is a random sample from gamma distribution:**

$$f(x;\alpha,\beta) = \frac{\beta^\alpha}{\Gamma(\alpha)}\, x^{\alpha-1}\, e^{-\beta x}\ ;\ x > 0$$

Derive the MME for parameters α and β.

$$X \sim \text{Gamma}\left(\alpha, \frac{1}{\beta}\right) \Rightarrow E(X) = \frac{\alpha}{\beta} \ \text{ and } V(X) = \frac{\alpha}{\beta^2}$$

$\mu_i' = M_i$, where $\mu_i' = E(X^i)$ and $M_i = \frac{1}{n}\sum_{j=1}^{n} X_j^i$ — p. 46

$\mu_1' = M_1$	$\mu_2' = M_2$
$E(X^1) = \frac{1}{n}\sum_{j=1}^{n} X_j^1$ $E(X) = \frac{1}{n}\sum_{j=1}^{n} X_j$ $\bar{X} = \frac{\alpha}{\beta}$ ... (1) $\alpha = \bar{X}\beta$ ... (2)	$E(X^2) = \frac{1}{n}\sum_{j=1}^{n} X_j^2$ $V(X) + (E(X))^2 = \frac{1}{n}\sum_{j=1}^{n} X_j^2$ $\frac{\alpha}{\beta^2} + \left(\frac{\alpha}{\beta}\right)^2 = \frac{1}{n}\sum_{j=1}^{n} X_j^2$ ... (3)

$V(X) = E(X^2) - (E(X))^2$

$V(X) + (E(X))^2 = E(X^2)$

From (2) and (3) we get

$$\frac{\bar{X}\beta}{\beta^2} + \left(\frac{\bar{X}\beta}{\beta}\right)^2 = \frac{1}{n}\sum_{j=1}^{n} X_j^2$$

$$\Rightarrow \frac{\bar{X}}{\beta} + \bar{X}^2 = \frac{1}{n}\sum_{j=1}^{n} X_j^2 \quad \Rightarrow \frac{\bar{X}}{\beta} = \frac{1}{n}\sum_{j=1}^{n} X_j^2 - \bar{X}^2$$

$$\Rightarrow \hat{\beta} = \frac{\bar{X}}{\frac{1}{n}\sum_{j=1}^{n} X_j^2 - \bar{X}^2}$$

Using (2) we get

$$\Rightarrow \frac{\hat{\alpha}}{\bar{X}} = \frac{\bar{X}}{\frac{1}{n}\sum_{j=1}^{n} X_j^2 - \bar{X}^2}$$

$$\Rightarrow \hat{\alpha} = \frac{\bar{X}^2}{\frac{1}{n}\sum_{j=1}^{n} X_j^2 - \bar{X}^2}$$

Note:	
• $\prod_{i=1}^{n} x_i^a = (\prod_{i=1}^{n} x_i)^a$ • $\prod_{i=1}^{n} e^{\frac{x_i}{\theta}} = e^{\frac{\sum_{i=1}^{n} x_i}{\theta}}$ • $\prod_{i=1}^{n} \frac{2}{\theta} = \left(\frac{2}{\theta}\right)^n$	• $\prod_{i=1}^{n} x_i! = \prod_{i=1}^{n} x_i!$ • $\prod_{i=1}^{n} e^{\theta} = e^{n\theta}$ • $ln(\prod_{i=1}^{n} x_i) = \sum_{i=1}^{n} ln\,(x_i)$ • $\sum_{i=1}^{n} 1 = n$

3.2: **Find the MME and the MLE for the parameter p of Bernoulli distribution:**

$$f(x) = p^x q^{1-x}, \; x = 0, 1$$

Then, determine the unbiasedness, sufficiency and consistency of the MLE

$\Rightarrow X \sim \text{Bernoulli}(p) \Rightarrow E(X) = p$ *and* $V(X) = pq$ were, $q = 1 - p$

MME:

$$\mu_1' = M_1 \Rightarrow E(X^1) = \frac{1}{n}\sum_{j=1}^{n} X_j^1 \Rightarrow \hat{p} = \overline{X}$$

$\mu_i' = M_i$, *where* $\mu_i' = E(X^i)$ *and* $M_i = \frac{1}{n}\sum_{j=1}^{n} X_j^i$ p.46

MLE:

(1) $L = \prod_{i=1}^{n} f(x_i) = \prod_{i=1}^{n} p^{x_i} q^{1-x_i} = p^{\sum_{i=1}^{n} x_i} q^{n-\sum_{i=1}^{n} x_i}$

(2) $\log L = \sum_{i=1}^{n} x_i \log p + (n - \sum_{i=1}^{n} x_i) \log(1-p)$

(3) $\frac{\partial}{\partial p} \log L = 0$ $\quad$ $\frac{\partial \log L(\theta_i; x)}{\partial \theta_i} = 0$ p.48

$$\Rightarrow \frac{\sum_{i=1}^{n} x_i}{p} - \frac{n - \sum_{i=1}^{n} x_i}{1-p} = 0$$

$$\Rightarrow \frac{(1-p)\sum_{i=1}^{n} x_i - p(n - \sum_{i=1}^{n} x_i)}{p(1-p)} = 0$$

$$\Rightarrow (1-p)\sum_{i=1}^{n} x_i - p(n - \sum_{i=1}^{n} x_i) = 0$$

$$\Rightarrow \sum_{i=1}^{n} x_i - p\sum_{i=1}^{n} x_i - np + p\sum_{i=1}^{n} x_i = 0$$

$$\Rightarrow \sum_{i=1}^{n} x_i - np = 0$$

$$\Rightarrow p = \frac{\sum_{i=1}^{n} x_i}{n}$$

$$\Rightarrow \boxed{\hat{p} = \bar{X}}$$

Unbiasedness:

$$E(\hat{p}) = E(\bar{X}) = E(X) = p$$

$\Rightarrow \hat{p} = \bar{X}$ *is unbiased estimator for p* $\quad$ $E(T) = \theta$ p.51

Sufficiency:

$\frac{\prod_{i=1}^{n} f(x_i,\theta)}{f_T(t,\theta)}$ *dose not depend on* θ $p.53$

By using factorization theorem

$\prod_{i=1}^{n} f(x_i,\theta) = K_1(t,\theta)K_2(x_1,x_2,\dots x_n)$ *where* K_2 *dose not depend on* θ $p.54$

We have $L = \prod_{i=1}^{n} f(x_i) = \prod_{i=1}^{n} p^{x_i} q^{1-x_i}$

$$= p^{\sum_{i=1}^{n} x_i} q^{n-\sum_{i=1}^{n} x_i}$$

$$= p^{n\bar{X}} q^{n-n\bar{X}}$$

We can write $\prod_{i=1}^{n} f(x_i) = K_1(\bar{X},P)K_2(x_1,x_2,\dots,x_n)$,

Where $K_2(x_1,x_2,\dots,x_n) = 1$

$\Rightarrow$ $\hat{p} = \bar{X}$ *sufficient of p*

Consistency:

$\lim_{n\to\infty} P(|T_n - \theta| \geq \varepsilon) = 0$ $p.53$

Theorem 3.2

1. $\lim_{n\to\infty} E(T_n) = \theta$
2. $\lim_{n\to\infty} V(T_n) = 0$

$p.53$

1. $\lim_{n\to\infty} E(T_n) = \theta$	2. $\lim_{n\to\infty} V(T_n) = 0$
$E(\hat{p}) = E(\bar{X}) = E(X) = p$ $\Rightarrow \lim_{n\to\infty} E(\hat{p}) = p \dots (1)$	$V(\hat{p}) = V(\bar{X}) = \frac{V(X)}{n} = \frac{p(1-p)}{n}$ $\Rightarrow \lim_{n\to\infty} V(\hat{p})$ $= \lim_{n\to\infty} \frac{p(1-p)}{n} = 0 \dots (2)$

Then, from (1) and (2) $\hat{p} = \bar{X}$ *consistent of P*

3.3: **Let $f(x) = \theta e^{-\theta x}, x > 0$, and let T be an estimator for $\tau(\theta)$. Study if T is unbiased, consistent estimator for $\tau(\theta)$, then compute MSE in the three cases**

(a) $T = \bar{X}$ and $\tau(\theta) = \frac{1}{\theta}$

Given that $X \sim \exp\left(\frac{1}{\theta}\right) \Rightarrow E(X) = \frac{1}{\theta}$ and $V(X) = \frac{1}{\theta^2}$

Unbiasedness:

$$E(T) = E(\bar{X}) = E(X) = \frac{1}{\theta} = \boldsymbol{\tau(\theta)}$$

$\Rightarrow T = \bar{X}$ *is unbiased estimator of* $\frac{1}{\theta}$ $\quad$ $\boxed{E(T) = \theta}$ $p.51$

Consistency:

Theorem 3.2 $p.53$
1. $\lim_{n\to\infty} E(T_n) = \theta$
2. $\lim_{n\to\infty} V(T_n) = 0$

1. $\lim_{n\to\infty} E(T_n) = \theta$	2. $\lim_{n\to\infty} V(T_n) = 0$
$E(T) = E(\bar{X}) = E(X) = \frac{1}{\theta}$ $\Rightarrow \lim_{n\to\infty} E(T) = \frac{1}{\theta} = \boldsymbol{\tau(\theta)}$	$V(T) = V(\bar{X}) = \frac{V(X)}{n} = \frac{\frac{1}{\theta^2}}{n} = \frac{1}{n\theta^2}$ $\Rightarrow \lim_{n\to\infty} V(T) = \lim_{n\to\infty} \frac{1}{n\theta^2} = 0$

MSE: $\quad$ $\boxed{MSE(T) = V(T) + [\theta - E(T)]^2}$ $p.52$

$MSE(T) = V(T) + \left[\frac{1}{\theta} - E(T)\right]^2$ $= \frac{1}{n\theta^2} + \left[\frac{1}{\theta} - \frac{1}{\theta}\right]^2 = \frac{1}{n\theta^2}$	Or Since $T = \bar{X}$ *is unbiased estimator of* $\frac{1}{\theta}$ then $MSE(T) = V(T) + [\theta - E(T)]^2$ $= V(T) + \underbrace{[\theta - \theta]^2}_{0}$ $= V(\bar{X}) = \frac{V(X)}{n} = \frac{\frac{1}{\theta^2}}{n} = \frac{1}{n\theta^2}$

(b) $T = \frac{1}{\bar{X}}$ **and** $\tau(\theta) = \theta$

Since $X \sim \exp\left(\frac{1}{\theta}\right) \equiv X \sim \text{Gamma}\left(1, \frac{1}{\theta}\right)$

$$E\left(\frac{1}{\bar{x}}\right) = E\left(\frac{n}{\sum X}\right) = nE\left(\frac{1}{\sum X}\right)$$

$\Rightarrow y = \sum_{i=1}^{n} X_i \sim \text{Gamma}\left(n, \frac{1}{\theta}\right) \Rightarrow f(y) = \frac{\theta^n}{\Gamma(n)} y^{n-1} e^{-\theta y}$ ⟶from Ch2

Then, $E\left(\frac{1}{Y}\right) = \int_0^\infty \frac{1}{y} f(y)dy = \frac{\theta^n}{\Gamma(n)} \int_0^\infty \frac{1}{y} y^{n-1} e^{-\theta y} dy$

$$= \frac{\theta^n}{\Gamma(n)} \int_0^\infty y^{n-2} e^{-\theta y} dy$$

$$= \frac{\theta^n}{\Gamma(n)} \frac{\Gamma(n-1)}{\theta^{n-1}} \underbrace{\int_0^\infty \frac{\theta^{n-1}}{\Gamma(n-1)} y^{n-2} e^{-\theta y} dy}_{1}$$

$$= \frac{\theta^n}{\Gamma(n)} \frac{\Gamma(n-1)}{\theta^{n-1}}$$

$$= \frac{\theta^n}{(n-1)\Gamma(n-1)} \frac{\Gamma(n-1)}{\theta^{n-1}} = \boxed{\frac{\theta}{n-1}}$$

$$\boxed{\Gamma(n) = (n-1)!}$$

$E\left(\frac{1}{Y^2}\right) = \int_0^\infty \frac{1}{y^2} f(y)dy = \frac{\theta^n}{\Gamma(n)} \int_0^\infty \frac{1}{y^2} y^{n-1} e^{-\theta y} dy$

$$= \frac{\theta^n}{\Gamma(n)} \int_0^\infty y^{n-3} e^{-\theta y} dy$$

$$= \frac{\theta^n}{\Gamma(n)} \frac{\Gamma(n-2)}{\theta^{n-2}} \underbrace{\int_0^\infty \frac{\theta^{n-2}}{\Gamma(n-2)} y^{n-3} e^{-\theta y} dy}_{1}$$

$$= \frac{\theta^n}{(n-1)(n-2)\Gamma(n-2)} \frac{\Gamma(n-2)}{\theta^{n-2}} = \frac{\theta^2}{(n-1)(n-2)}$$

Then, $V\left(\frac{1}{Y}\right) = E\left(\frac{1}{Y^2}\right) - \left(E\left(\frac{1}{Y}\right)\right)^2$

$$= \frac{\theta^2}{(n-1)(n-2)} - \left(\frac{\theta}{n-1}\right)^2 = \boxed{\frac{\theta^2}{(n-1)^2(n-2)}}$$

$$E\left(\frac{1}{\bar{X}}\right) \neq \frac{E(1)}{E(\bar{X})}$$

$$E\left(\frac{X}{a}\right) = \frac{E(X)}{a}$$

Unbiasedness: $\boxed{E(T) = \theta}$ $p.51$

$$E(T) = E\left(\frac{1}{\bar{X}}\right) = nE\left(\frac{1}{\sum_{i=1}^{n} X_i}\right) = nE\left(\frac{1}{Y}\right) = \frac{n\theta}{\mathrm{n-1}} \neq \theta = \boldsymbol{\tau(\theta)}$$

$T = \frac{1}{\bar{X}}$ *is a biased estimator of* $\boldsymbol{\tau(\theta)} = \theta.$

Consistency:

$\boxed{\begin{array}{l} \text{Theorem } 3.2 \\ 1.\ \lim_{n\to\infty} E(T_n) = \theta \\ 2.\ \lim_{n\to\infty} V(T_n) = 0 \end{array}}$ $p.53$

$1.\ \lim_{n\to\infty} E(T_n) = \theta$	$2.\ \lim_{n\to\infty} V(T_n) = 0$
$E(T) = E\left(\frac{1}{\bar{X}}\right) = nE\left(\frac{1}{\sum_{i=1}^{n} X_i}\right) = nE\left(\frac{1}{Y}\right) = \frac{n\theta}{\mathrm{n-1}}$ $\Rightarrow \lim_{n\to\infty} E(T) = \lim_{n\to\infty} \frac{n\theta}{\mathrm{n-1}} = \theta = \boldsymbol{\tau(\theta)} \ldots (1)$	$V(T) = V\left(\frac{1}{\bar{X}}\right) = n^2 V\left(\frac{1}{\sum_{i=1}^{n} X_i}\right)$ $= n^2 V\left(\frac{1}{Y}\right) = \frac{n^2\theta^2}{(\mathrm{n-1})^2(\mathrm{n-2})}$ $\Rightarrow \lim_{n\to\infty} V(T) = \lim_{n\to\infty} \frac{n^2\theta^2}{(\mathrm{n-1})^2(\mathrm{n-2})} = 0 \ldots (2)$

Then, from (1) and (2) $T = \frac{1}{\bar{X}}$ *is consistent. of* θ.

MSE: $\boxed{MSE(T) = V(T) + [\theta - E(T)]^2}$ $p.52$

$$MSE(T) = V(T) + [\theta - E(T)]^2$$

$$= \frac{n^2\theta^2}{(\mathrm{n-1})^2(\mathrm{n-2})} + \left[\theta - \frac{n\theta}{\mathrm{n-1}}\right]^2 = \boxed{\frac{(n^2+n-2)\theta^2}{(\mathrm{n-1})^2(\mathrm{n-2})}}$$

(c) $\boldsymbol{T = \frac{n-1}{\sum_{i=1}^{n} X_i}}$ **and** $\boldsymbol{\tau(\theta) = \theta}$

Unbiasedness: $\boxed{E(T) = \theta}$ $p.51$

$$E(T) = E\left(\frac{n-1}{\sum_{i=1}^{n} X_i}\right) = (n-1)E\left(\frac{1}{Y}\right)$$

$$= (n-1)\frac{\theta}{\text{n}-1} = \theta$$

$$T = \frac{n-1}{\sum_{i=1}^{n} X_i} \textit{ is an unbiased estimator of } \tau(\theta) = \theta.$$

Consistency:

Theorem 3.2
1. $\lim_{n\to\infty} E(T_n) = \theta$
2. $\lim_{n\to\infty} V(T_n) = 0$

$p.53$

1. $\lim_{n\to\infty} E(T_n) = \theta$	2. $\lim_{n\to\infty} V(T_n) = 0$
$E(T) = E\left(\frac{n-1}{\sum_{i=1}^{n} X_i}\right) = (n-1)E\left(\frac{1}{Y}\right) = \theta$ $\Rightarrow \lim_{n\to\infty} E(T) = \lim_{n\to\infty} \theta = \theta \dots (1)$	$V(T) = V\left(\frac{n-1}{\sum_{i=1}^{n} X_i}\right) = (n-1)^2 V\left(\frac{1}{Y}\right)$ $= (n-1)^2 \frac{\theta^2}{(\text{n}-1)^2(\text{n}-2)} = \frac{\theta^2}{(\text{n}-2)}$ $\Rightarrow \lim_{n\to\infty} V(T) = \lim_{n\to\infty} \frac{\theta^2}{(\text{n}-2)} = 0 \dots (2)$

Then, from (1) and (2) $T = \frac{n-1}{\sum_{i=1}^{n} X_i}$ is consistent. *of* θ.

MSE: $\boxed{MSE(T) = V(T) + [\theta - E(T)]^2}$ $p.52$

$$MSE(T) = V(T) + [\theta - E(T)]^2$$

$$= \frac{\theta^2}{(\text{n}-2)} + [\theta - \theta]^2 = \frac{\theta^2}{(\text{n}-2)}$$

3.4: **If $X_1, X_2, \dots, X_n$ be a random sample from $(x;\theta)$. Show if the given statistic T is sufficient statistic for θ:**

$$f(x;\theta) = e^{-(x-\theta)}\,; x > \theta \quad ; \quad T = Y_1 = \text{Minimum}\,(X_1, X_2, \dots, X_n).$$

$\dfrac{\prod_{i=1}^{n} f(x_i,\theta)}{f_T(t,\theta)}$ *dose not depend on* θ $p.53$

$$\prod_{i=1}^{n} f_X(x_i) = \prod_{i=1}^{n} e^{-(x-\theta)} = e^{-(\sum_{i=1}^{n} x - n\theta)} = e^{-\sum_{i=1}^{n} x + n\theta}$$

$$f_T(t) = f_{Y_1}(y_1) = n f_X(y_1)[1 - F_X(y_1)]^{n-1} \quad , \quad y_1 > \theta$$

$$\begin{aligned} F_X(x) = P(X \le x) &= \int_{\theta}^{x} f_X(t)\,dt \\ &= \int_{\theta}^{x} e^{-(t-\theta)}\,dt \\ &= \left[-e^{-(t-\theta)}\right]_{\theta}^{x} \\ &= 1 - e^{-(x-\theta)} \quad , x > \theta \end{aligned}$$

$$\begin{aligned} f_T(t) = f_{Y_1}(y_1) &= n e^{-(y_1-\theta)}\left[1 - \left(1 - e^{-(y_1-\theta)}\right)\right]^{n-1} \\ &= n e^{-(y_1-\theta)}\left[e^{-(y_1-\theta)}\right]^{n-1} \\ &= n e^{-(y_1-\theta)}\, e^{-(y_1-\theta)^{n}} e^{-(y_1-\theta)^{-1}} \\ &= n e^{-n(y_1-\theta)} \quad , \quad y_1 > \theta \end{aligned}$$

$$f(X_1, X_2, \dots, X_n) = \frac{\prod_{i=1}^{n} f_X(x_i)}{f_T(t)} = \frac{e^{-\sum_{i=1}^{n} x_i + n\theta}}{n e^{-n(y_1-\theta)}}$$

$$= \frac{e^{-\sum_{i=1}^{n} x_i}\ e^{n\theta}}{n e^{-n y_1}\ \ e^{n\theta}} = \frac{e^{-\sum_{i=1}^{n} x_i}}{n e^{-n y_1}}$$

Which does not depend on θ, then $T = min(X_1, X_2, \dots, X_n) = Y_1$ is sufficient statistic of θ

3.5: **Suppose for a given random variable T_1 and T_2 be two independents unbiased estimators for θ and with the same variance σ^2. Define two random variables as**

$$Y = \frac{3T_1+2T_2}{5} \text{ and } Z = \frac{T_1+2T_2}{3}$$

Find MSE (Y) and $MSE(Z)$ and compare between them.

Since T_1 and T_2 are unbiased estimators of θ

Then, $E(T_1) = \theta$ and $E(T_2) = \theta$

$MSE(Y) = V(Y) + [\theta - E(Y)]^2$	$MSE(Z) = V(Z) + [\theta - E(Z)]^2$
$E(Y) = E\left(\frac{3T_1+2T_2}{5}\right)$	$E(Z) = E\left(\frac{T_1+2T_2}{3}\right)$
$= \frac{3}{5}E(T_1) + \frac{2}{5}E(T_2)$	$= \frac{1}{3}E(T_1) + \frac{2}{3}E(T_2)$
$= \frac{3}{5}\theta + \frac{2}{5}\theta = \theta$	$= \frac{1}{3}\theta + \frac{2}{3}\theta = \theta$
$\Rightarrow Y$ is an unbiased estimator of θ	$\Rightarrow Z$ is an unbiased estimator of θ
$\Rightarrow MSE(Y) = V(Y)$	$\Rightarrow MSE(Z) = V(Z)$
$V(Y) = V\left(\frac{3T_1+2T_2}{5}\right)$	$V(Z) = V\left(\frac{T_1+2T_2}{3}\right)$
$= \frac{9}{25}V(T_1) + \frac{4}{25}V(T_2)$	$= \frac{1}{9}V(T_1) + \frac{4}{9}V(T_2)$
$= \frac{9}{25}\sigma^2 + \frac{4}{25}\sigma^2 = \frac{13}{25}\sigma^2$	$= \frac{1}{9}\sigma^2 + \frac{4}{9}\sigma^2 = \frac{5}{9}\sigma^2$
$\Rightarrow MSE(Y) = \frac{13}{25}\sigma^2$	$\Rightarrow MSE(Z) = \frac{5}{9}\sigma^2$
$= 0.52\sigma^2$	$= 0.56\sigma^2$

Comparing $MSE(Y)$ and $MSE(Z)$

$0.52\,\sigma^2 < 0.56\,\sigma^2$ **Y** is better estimator of θ than Z

3.6: **Let $f(x,\theta) = \frac{1}{\theta}$; $x \in (0,\theta)$ and let T be an estimator for θ. Study if T is unbiased, consistent and compute MSE, then compare between their variances for the following cases:**

Given $X \sim Uniform(0,\theta)$

$$\Rightarrow f_X(x) = \frac{1}{\theta},\ E(X) = \frac{\theta}{2}\ , V(X) = \frac{\theta^2}{12} \text{ and } F_X(x) = \begin{cases} 0, & x < 0 \\ \frac{x}{\theta}, & 0 \le x < \theta \\ 1, & x \ge \theta \end{cases}$$

(a) $T = min(X_1, X_2, \dots, X_n) = Y_1$

$$f_{Y_1}(y_1) = n f_X(y_1)[1 - F_X(y_1)]^{n-1}$$

$$= n\frac{1}{\theta}\left[1 - \frac{y_1}{\theta}\right]^{n-1} \quad ; \ 0 \le y_1 < \theta$$

$E(Y_1) = \int_0^\theta y_1 f_{Y_1}(y_1)d\,y_1$ $= \int_0^\theta n\frac{y_1}{\theta}\left[1 - \frac{y_1}{\theta}\right]^{n-1} d\,y_1$, let $u = \frac{y_1}{\theta} \Rightarrow du = \frac{1}{\theta}dy_1 \Rightarrow \theta du = dy_1$ $= n\theta \int_0^1 u^{2-1}[1-u]^{n-1}d\,u$ $\beta(a,b) = \int_0^1 x^{a-1}[1-x]^{b-1}d\,x = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$ $= n\theta\beta(2,n) = n\theta\frac{\Gamma(2)\Gamma(n)}{\Gamma(2+n)}$ $= n\theta\frac{\Gamma(n)}{(n+1)n\Gamma(n)}$ $= \frac{\theta}{(n+1)}$	$E(Y_1^2) = \int_0^\theta y_1^2 f_{Y_1}(y_1)d\,y_1$ $= \int_0^\theta n\frac{y_1^2}{\theta}\left[1 - \frac{y_1}{\theta}\right]^{n-1} d\,y_1$ $= \int_0^\theta \theta\theta n\frac{{y_1}^2}{\theta\theta}\left[1 - \frac{y_1}{\theta}\right]^{n-1} d\,y_1$ $= \int_0^\theta \theta^2 n\left(\frac{y_1}{\theta}\right)^2\left[1 - \frac{y_1}{\theta}\right]^{n-1} d\,y_1$, let $u = \frac{y_1}{\theta} \Rightarrow du = \frac{1}{\theta}dy_1 \Rightarrow \theta du = dy_1$ $= n\theta^2\int_0^1 u^{3-1}[1-u]^{n-1}d\,y_1$ $= n\theta^2\beta(3,n) = n\theta^2\frac{\Gamma(3)\Gamma(n)}{\Gamma(3+n)}$ $= n\theta^2\frac{\Gamma(n)}{(n+2)(n+1)n\Gamma(n)}$ $= \frac{2\theta^2}{(n+2)(n+1)}$

$$V(Y_1) = E(Y_1^2) - \left(E(Y_1)\right)^2$$

$$= \frac{2\theta^2}{(n+2)(n+1)} - \frac{\theta^2}{(n+1)^2} = \frac{n\theta^2}{(n+2)(n+1)^2}$$

Unbiasedness:

$E(\mathrm{T}) = E(Y_1) = \frac{\theta}{n+1}$

$\Rightarrow T = Y_1$ *is a biased estimator of* θ

Consistency:

Theorem 3.2
1. $\lim_{n\to\infty} E(T_n) = \theta$
2. $\lim_{n\to\infty} V(T_n) = 0$

$p. 53$

(1) $E(\mathrm{T}) = E(Y_1) = \frac{\theta}{n+1}$

$\Rightarrow \lim_{n\to\infty} E(T) = \lim_{n\to\infty} \frac{\theta}{n+1} = 0 \neq \theta$ (it's not asymptotically unbiased)

No need to check the other condition we can see here that $T = Y_1$ is not a consistent estimator of θ

MSE:

$MSE(T) = V(T) + [\theta - E(T)]^2$ $p. 52$

$$MSE(T) = V(T) + [\theta - E(T)]^2$$

$$= \frac{n\theta^2}{(n+1)^2(n+2)} + \left[\theta - \frac{\theta}{n+1}\right]^2$$

$$= \frac{(n+2n^2+n^3)\theta^2}{(n+1)^2(n+2)}$$

(b) $\boldsymbol{T = nY_1}$

Unbiasedness:

$$E(\mathrm{T}) = E(nY_1) = \frac{n\theta}{n+1}$$

$$\Rightarrow \quad T = nY_1 \ is\ a\ biased\ estimator\ of\ \ \theta$$

Consistency:

Theorem 3.2
1. $\lim_{n\to\infty} E(T_n) = \theta$
2. $\lim_{n\to\infty} V(T_n) = 0$

$p. 53$

1. $\lim_{n\to\infty} E(T_n) = \theta$	2. $\lim_{n\to\infty} V(T_n) = 0$
$E(\mathrm{T}) = E(nY_1) = \frac{n\theta}{n+1}$ $\Rightarrow \lim_{n\to\infty} E(nY_1)$ $\Rightarrow \lim_{n\to\infty} \frac{n\theta}{n+1} = \theta$ (it's asymptotically unbiased)	$V(T) = V(nY_1\ ) = n^2 V(Y_1)$ $= n^2 \frac{n\theta^2}{(n+1)^2(\mathrm{n}+2)} = \frac{n^3\theta^2}{(n+1)^2(\mathrm{n}+2)}$ $\Rightarrow \lim_{n\to\infty} V(T)$ $\Rightarrow \lim_{n\to\infty} \frac{n^3\theta^2}{(n+1)^2(\mathrm{n}+2)} = \theta^2 \neq 0$

Then, we can see here that $T = nY_1$ is not a consistent estimator $of\ \ \theta$

MSE:

$MSE(T) = V(T) + [\theta - E(T)]^2$ $p. 52$

$$MSE(T) = V(T) + [\theta - E(T)]^2$$

$$= \frac{n^3\theta^2}{(n+1)^2(\mathrm{n}+2)} + \left[\theta - \frac{n\theta}{n+1}\right]^2$$

$$= \frac{(n^3+n+2)\theta^2}{(n+1)^2(\mathrm{n}+2)}$$

(c) $\boldsymbol{T = 2\bar{X}}$

Unbiasedness:

$$E(\mathrm{T}) = E(2\bar{X}) = 2E(\bar{X}) = 2E(X) = 2\frac{\theta}{2} = \theta$$

$\Rightarrow \; T = 2\bar{X}$ *is an unbiased estimator of* θ

Consistency:

Theorem 3.2
1. $\lim_{n\to\infty} E(T_n) = \theta$
2. $\lim_{n\to\infty} V(T_n) = 0$

p. 53

1. $\lim_{n\to\infty} E(T_n) = \theta$	2. $\lim_{n\to\infty} V(T_n) = 0$
$E(\mathrm{T}) = E(2\bar{X}) = 2E(\bar{X})$ $= 2E(X) = 2\frac{\theta}{2} = \theta$ $\Rightarrow \lim_{n\to\infty} E(2\bar{X})$ $\Rightarrow \lim_{n\to\infty} \theta = \theta$	$V(T) = V(2\bar{X}) = 4V(\bar{X})$ $= \frac{4V(X)}{n} = \frac{4\left(\frac{\theta^2}{12}\right)}{n} = \frac{\boldsymbol{\theta^2}}{\boldsymbol{3n}}$ $\Rightarrow \lim_{n\to\infty} V(T)$ $\Rightarrow \lim_{n\to\infty} \frac{\boldsymbol{\theta^2}}{\boldsymbol{3n}} = \boldsymbol{0}$

Then, we can see here that $T = 2\bar{X}_1$ is a consistent estimator of θ .

MSE:

$MSE(T) = V(T) + [\theta - E(T)]^2$ p. 52

since T is an unbiased estimator of θ

$$\Rightarrow \; MSE(T) = V(T) = \frac{\boldsymbol{\theta^2}}{\boldsymbol{3n}}.$$

(d) $\boldsymbol{T = \frac{n+1}{n} Y_n}$ **,** where $Y_n = maximum$

$$f_{Y_n}(y_n) = n f_X(y_n)[F_X(y_n)]^{n-1},$$

$$= n \frac{1}{\theta} \left[\frac{y_n}{\theta}\right]^{n-1}$$

$$= \frac{n}{\theta^n} {y_n}^{n-1} \; ; \quad 0 \le y_n < \theta$$

$E(Y_n) = \int_0^\theta y_n f_{Y_n}(y_n) d\, y_n$ $= \int_0^\theta n \frac{y_n}{\theta} \left[\frac{y_n}{\theta}\right]^{n-1} d\, y_n$ $= \int_0^\theta \frac{n}{\theta^n} {y_n}^n d\, y_n = \frac{n\theta}{n+1}$	$E(Y_n^2) = \int_0^\theta {y_n}^2 f_{Y_n}(y_n)\, dy_n$ $= \int_0^\theta n \frac{{y_n}^2}{\theta} \left[\frac{y_n}{\theta}\right]^{n-1} dy_n$ $= \int_0^\theta \frac{n}{\theta^n} {y_n}^{n+1} d\, y_n = \frac{\theta^2 n}{n+2}$

$$V(Y_n) = E\left({Y_n}^2\right) - \left(E(Y_n)\right)^2$$

$$= \frac{\theta^2 n}{n+2} - \left(\frac{n\theta}{n+1}\right)^2 = \frac{n\theta^2}{(n+1)^2(n+2)}$$

<u>Unbiasedness:</u>

$$E(\mathrm{T}) = E\left(\frac{n+1}{n} Y_n\right) = \frac{n+1}{n} E(Y_n) = \frac{n+1}{n}\left(\frac{\boldsymbol{\theta n}}{\boldsymbol{n+1}}\right) = \theta$$

$\Rightarrow T = \frac{n+1}{n} Y_n$ *is an unbiased estimator of* θ

<u>Consistency</u>:

Theorem 3.2
1. $\lim_{n\to\infty} E(T_n) = \theta$
2. $\lim_{n\to\infty} V(T_n) = 0$

$p.53$

1. $\lim_{n\to\infty} E(T_n) = \theta$	2. $\lim_{n\to\infty} V(T_n) = 0$
$E(\mathrm{T}) = E\left(\frac{n+1}{n} Y_n\right) = \frac{n+1}{n} E(Y_n)$ $= \frac{n+1}{n}\left(\frac{\theta n}{n+1}\right) = \theta$ $\Rightarrow \lim_{n\to\infty} E(\mathrm{T}) = \lim_{n\to\infty} \theta = \theta$	$V(T) = V\left(\frac{n+1}{n} Y_n\right) = \left(\frac{n+1}{n}\right)^2 V(Y_n)$ $= \frac{(n+1)^2}{n^2} \frac{n\theta^2}{(n+1)^2(n+2)} = \frac{\theta^2}{n(n+2)}$ $\Rightarrow \lim_{n\to\infty} V(T) = \lim_{n\to\infty} \frac{\theta^2}{n(n+2)} = 0$

Then, we can see here that $T = \frac{n+1}{n} Y_n$ is a consistent estimator of θ .

MSE: $\boxed{MSE(T) = V(T) + [\theta - E(T)]^2}$ $p.52$

since T is an unbiased estimator of θ

$$MSE(T) = V(T) = \frac{\theta^2}{n(n+2)}$$

Comparing the MSE

We will compare (c) and (d) because they are unbiased estimators

$MSE(2\bar{X}) = \frac{\theta^2}{3n}$ $= \frac{(n+2)\theta^2}{3n(n+2)}$	$MSE\left(\frac{n+1}{n} Y_n\right) = \frac{\theta^2}{n(n+2)}$ $= \frac{3\theta^2}{3n(n+2)}$

Since n=1,2,3, …

$$n \geq 1$$

$$\Rightarrow \quad n + 2 \geq 1 + 2$$

$$\Rightarrow \quad n + 2 \geq 3$$

$$\Rightarrow \quad \frac{(n+2)\theta^2}{3n(n+2)} \geq \frac{3\theta^2}{3n(n+2)}$$

$\Rightarrow \frac{n+1}{n} Y_n$ is a better estimator of θ than $2\bar{X}$

3.7: **For a random sample $X_1, X_2, \dots, X_n$ drawn from the following distributions, find the Fisher information, $I_X(\theta)$:**

$I_X(\theta) = nI(\theta)$

$$I(\theta) = E\left[\frac{d}{d\theta} lnf(x)\right]^2 = -E\left[\frac{d^2}{d\theta^2} lnf(x)\right] \quad p.73$$

(a) Bernoulli (θ)

$$f(x) = \theta^x(1-\theta)^{1-x}, \quad x = 0,1 \qquad E(X) = \theta \quad V(X) = \theta(1-\theta)$$

$$lnf(x) = \ln[\theta^x(1-\theta)^{1-x}]$$

$$= xln\theta + (1-x)\ln(1-\theta) \qquad \ln(ab) = \ln(a) + \ln(b)$$

$$\frac{d}{d\theta} lnf(x) = \frac{x}{\theta} - \frac{1-x}{(1-\theta)} \quad \dots (1)$$

$$\frac{d^2}{d\theta^2} lnf(x) = \frac{-x}{\theta^2} - (-(-1))\frac{(1-x)}{(1-\theta)^2}$$

$$= \frac{-x}{\theta^2} - \frac{(1-x)}{(1-\theta)^2} \dots (2)$$

$$I(\theta) = E\left[\frac{d}{d\theta} lnf(x)\right]^2 = E\left[\frac{x}{\theta} - \frac{1-x}{(1-\theta)}\right]^2 \qquad \text{from (1)}$$

$$= E\left[\frac{x(1-\theta) - \theta(1-x)}{\theta(1-\theta)}\right]^2$$

$$= \frac{1}{\theta^2(1-\theta)^2} E[x - \theta x - \theta + \theta x]^2$$

$$= \frac{1}{\theta^2(1-\theta)^2} E[X-\theta]^2 \qquad E(X) = \theta \quad V(X) = E[X - E(X)]^2$$

$$= \frac{1}{\theta^2(1-\theta)^2} V[X]$$

$$= \frac{1}{\theta^2(1-\theta)^2} \theta(1-\theta)$$

$$= \frac{1}{\theta(1-\theta)}$$

$$\Rightarrow I_X(\theta) = nI(\theta) = \frac{n}{\theta(1-\theta)}$$

or

$$I(\theta) = -E\left[\frac{d^2}{d\theta^2} \ln f(x)\right] = -E\left[\frac{-X}{\theta^2} - \frac{(1-X)}{(1-\theta)^2}\right] \quad \text{from (2)}$$

$$= E\left[\frac{X}{\theta^2} + \frac{(1-X)}{(1-\theta)^2}\right]$$

$$= E\left[\frac{X(1-\theta)^2 + (1-X)\theta^2}{\theta^2(1-\theta)^2}\right]$$

$$= \frac{1}{\theta^2(1-\theta)^2} E[X - 2\theta X + X\theta^2 + \theta^2 - X\theta^2]$$

$$= \frac{1}{\theta^2(1-\theta)^2} E[X - 2\theta X + \theta^2]$$

$$= \frac{1}{\theta^2(1-\theta)^2} [E(X) - 2\theta E(X) + E(\theta^2)]$$

$$= \frac{1}{\theta^2(1-\theta)^2} [\theta - 2\theta\theta + \theta^2]$$

$$= \frac{1}{\theta^2(1-\theta)^2} \theta[1 - \theta]$$

$$= \frac{1}{\theta(1-\theta)}$$

$$\Rightarrow I_X(\theta) = nI(\theta) = \frac{n}{\theta(1-\theta)}$$

(b) Exponential $\left(\frac{1}{\theta}\right)$

$f(x) = \theta e^{-\theta x}, \quad x > 0$	$E(X) = \frac{1}{\theta}$ $V(X) = \frac{1}{\theta^2}$

$$ln f(x) = ln\theta + ln e^{-\theta x}$$

$$= ln\theta - \theta x$$

$$\frac{d}{d\theta} ln f(x) = \frac{1}{\theta} - x \ldots (1)$$

$$\frac{d^2}{d\theta^2} ln f(x) = \frac{-1}{\theta^2} \ldots (2)$$

or

$I(\theta) = E\left[\frac{d}{d\theta} ln f(x)\right]^2$	$I(\theta) = -E\left[\frac{d^2}{d\theta^2} ln f(x)\right]$
$= E\left[\frac{1}{\theta} - X\right]^2$ $= E\left[X - \frac{1}{\theta}\right]^2$ $= E[X - E(X)]^2$ $= V[X] = \frac{1}{\theta^2}$ $\Rightarrow I_X(\theta) = nI(\theta) = \frac{n}{\theta^2}$	$= -E\left[\frac{-1}{\theta^2}\right]$ $= -\left[\frac{-1}{\theta^2}\right]$ $= \frac{1}{\theta^2}$ $\Rightarrow I_X(\theta) = nI(\theta) = \frac{n}{\theta^2}$

(c) $\boldsymbol{Normal(\theta, \sigma^2)}$ **when** $\boldsymbol{\sigma^2\ known}$

$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}\frac{(x-\theta)^2}{\sigma^2}}, \quad x \in (-\infty, \infty)$	$E(X) = \theta$ $V(X) = \sigma^2$

$$f(x) = \frac{1}{(2\pi\sigma^2)^{1/2}} e^{-\frac{1}{2}\frac{(x-\theta)^2}{\sigma^2}}$$

$$lnf(x) = \ln(1) - \frac{1}{2}\ln(2\pi\sigma^2) - \frac{1}{2}\frac{(x-\theta)^2}{\sigma^2}$$

$$\ln\left(\frac{a}{b}\right) = \ln(a) - \ln(b)$$

$$lnf(x) = -\frac{1}{2}\ln(2\pi\sigma^2) - \frac{1}{2}\frac{(x-\theta)^2}{\sigma^2}$$

$$\frac{d}{d\theta} lnf(x) = \frac{-1}{2\sigma^2} 2(x-\theta)(-1) = \frac{(x-\theta)}{\sigma^2} \ldots (1)$$

$$\frac{d^2}{d\theta^2} lnf(x) = \frac{-1}{\sigma^2} \ldots (2)$$

or

$I(\theta) = E\left[\frac{d}{d\theta} lnf(x)\right]^2$	$I(\theta) = -E\left[\frac{d^2}{d\theta^2} lnf(x)\right]$
$= E\left[\frac{X-\theta}{\sigma^2}\right]^2$ $= \frac{1}{\sigma^4} E[X-\theta]^2$ $= \frac{1}{\sigma^4} E[X-E(X)]^2$ $= \frac{1}{\sigma^4} V[X] = \frac{\sigma^2}{\sigma^4} = \frac{1}{\sigma^2}$ $\Rightarrow I_X(\theta) = nI(\theta) = \frac{n}{\sigma^2}$	$= -E\left[\frac{-1}{\sigma^2}\right]$ $= -\left[\frac{-1}{\sigma^2}\right] = \frac{1}{\sigma^2}$ $\Rightarrow I_X(\theta) = nI(\theta) = \frac{n}{\sigma^2}$

3.8: **Let $X_1, X_2, \dots, X_n$ be a random sample drawn $N(\mu, \sigma^2)$, σ^2 is known. Find:**

(a) CRLB

$Var(T) \geq CRLB = \frac{((\tau(\theta)')^2}{nI(\theta)}$ p. 78

(i) $\tau(\mu) = \mu$	(ii) $\tau(\mu) = e^{\mu}$	(iii) $\tau(\mu) = \frac{1}{(\mu+1)}$
$\Rightarrow \tau(\mu)' = 1$ $\Rightarrow ((\tau(\mu)')^2 = 1$ $\Rightarrow CRLB = \frac{((\tau(\mu)')^2}{nI(\mu)}$ $nI(\mu) = \frac{n}{\sigma^2}$ from 3.7 (c) $= \frac{1}{\frac{n}{\sigma^2}}$ $= \frac{\sigma^2}{n}$	$\Rightarrow \tau(\mu)' = e^{\mu}$ $\Rightarrow ((\tau(\mu)')^2 = e^{2\mu}$ $\Rightarrow CRLB = \frac{((\tau(\mu)')^2}{nI(\mu)}$ $= \frac{e^{2\mu}}{\frac{n}{\sigma^2}}$ $= \frac{e^{2\mu}\sigma^2}{n}$	$\Rightarrow \tau(\mu)' = \frac{-1}{(\mu+1)^2}$ $\Rightarrow ((\tau(\mu)')^2 = \frac{1}{(\mu+1)^4}$ $\Rightarrow CRLB = \frac{((\tau(\mu)')^2}{nI(\mu)}$ $= \frac{\frac{1}{(\mu+1)^4}}{\frac{n}{\sigma^2}}$ $= \frac{\sigma^2}{n(\mu+1)^4}$

(b) MVUE of μ

T is called MVUE if
1. T unbiased for $\tau(\theta)$ $E(T) = \tau(\theta)$
2. $Var(T) = CRLB$

p. 77

We have seen that $\hat{\mu}_{MLE} = \hat{\mu}_{MME} = \bar{X}$

$E(\bar{X}) = E(X) = \mu \Rightarrow \bar{X}$ is an unbiased estimator of μ	$V(\bar{X}) = \frac{V(X)}{n} = \frac{\sigma^2}{n}$ $CRLB = \frac{\sigma^2}{n}$

Here we have $V(\bar{X}) = \frac{\sigma^2}{n}$ and $CRLB = \frac{\sigma^2}{n}$ (from 3.8 (a) (i))

$V(\bar{X}) = CRLB \Rightarrow \bar{X}$ is MVUE

3.10: **Let $X_1, X_2, \dots, X_n$ be a random sample from a distribution with pdf**

$$f(x\,;\,\theta) = \theta^2 x e^{-x\theta}, x > 0 \quad, \theta > 0 \qquad \Rightarrow f(x) = \frac{\theta^2 x^{2-1} e^{-x\theta}}{\Gamma(2)} \Rightarrow X \sim Gamma(2, \tfrac{1}{\theta})$$

(a) Argue that $Y = \sum_{i=1}^{n} x_i$ is a complete sufficient statistic for θ.

$f(x) = \theta^2 x e^{-x\theta}$ is a member of exponential family

if $f(x) = a(\theta)\, b(x) e^{c(\theta) d(x)}$, then $T = \sum d(x_i)$ is complete minimal sufficient if p. 69

$a(\theta) = \theta^2,\ b(x) = x\ , c(\theta) = -\theta\ ,\ d(x) = x$

Then, $Y = \sum_{i=1}^{n} d(x_i) = \sum_{i=1}^{n} x_i$ is a complete sufficient for θ

(b) Compute $E\left(\frac{1}{Y}\right)$ and find the function of Y which is the unique MVUE of θ.

$$E\left(\frac{1}{Y}\right) = E\left(\frac{1}{\sum_{i=1}^{n} x_i}\right)$$

We have $X \sim Gamma\left(2, \frac{1}{\theta}\right) \Rightarrow y = \sum_{i=1}^{n} x_i \sim \text{Gamma}(2n, \frac{1}{\theta})$

$$f(y) = \frac{\theta^{2n} y^{2n-1} e^{-y\theta}}{\Gamma(2n)}$$

$$E\left(\frac{1}{Y}\right) = \int_0^\infty \frac{1}{y} f(y) dy = \int_0^\infty \frac{1}{y} \frac{\theta^{2n} y^{2n-1} e^{-y\theta}}{\Gamma(2n)} dy$$

$$= \int_0^\infty \frac{\theta^{2n} y^{2n-2} e^{-y\theta}}{\Gamma(2n)} dy = \frac{\theta^{2n}}{\Gamma(2n)} \int_0^\infty y^{(2n-1)-1} e^{-y\theta} dy$$

$$= \frac{\theta^{2n}}{\Gamma(2n)} \frac{\Gamma(2n-1)}{\theta^{2n-1}} \underbrace{\int_0^\infty \frac{\theta^{2n-1}}{\Gamma(2n-1)} y^{(2n-1)-1} e^{-y\theta} dy}_{1}$$

$$= \frac{\theta^{2n}}{(2n-1)\Gamma(2n-1)} \frac{\Gamma(2n-1)}{\theta^{2n-1}} = \boxed{\frac{\theta}{2n-1}}$$

From (a) we got that $Y = \sum_{i=1}^{n} x_i$ is a complete sufficient for θ

We have to find a function of Y which is unbiased estimators $E\big(\tau(Y)\big) = \theta$

$$\Rightarrow E\left(\frac{1}{Y}\right) = \frac{\theta}{2n-1}$$

by using Lehman – Scheffe theorm p. 89

$$\Rightarrow E\left(\frac{2n-1}{Y}\right) = (2n-1) E\left(\frac{1}{Y}\right) = (2n-1)\frac{\theta}{2n-1} = \theta$$

Then $\frac{2n-1}{Y}$ is MVUE of θ

(c) Drive the MLE of θ and find the approximate distribution of it.

$$f(x\,;\,\theta) = \theta^2 x e^{-x\theta}$$

(1) $L = \prod_{i=1}^{n} f(x_i) = \prod_{i=1}^{n} \theta^2 x_i e^{-x_i\theta} = \theta^{2n} \prod_{i=1}^{n} x_i\, e^{-\theta\sum_{i=1}^{n} x_i}$

(2) $logL = 2nlog\theta + \sum_{i=1}^{n} \log(x_i) - \theta\sum_{i=1}^{n} x_i$

(3) $\frac{\partial}{\partial p} logL = 0$ $\qquad \boxed{\frac{\partial \log L(\theta_i;x)}{\partial\theta_i} = 0}$ $p.48$

$$\Rightarrow \frac{2n}{\theta} - \sum_{i=1}^{n} x_i = 0 \Rightarrow \hat{\theta} = \frac{2n}{\sum_{i=1}^{n} x_i} = \frac{2}{\bar{X}} \Rightarrow \hat{\theta} = \frac{2}{\bar{X}}$$

Suppose than $n \to \infty$ if $\hat{\tau}(\theta)$ be the MLE of $\tau(\theta)$, then $\hat{\tau}(\theta)$ has distribution as

$$\sqrt{n\big(\hat{\tau}(\theta) - \tau(\theta)\big)} \to N\left(0\,, \frac{\left((\tau(\theta)'\right)^2}{I_X(\theta)}\right) \quad or \quad \hat{\tau}(\theta) \to N\left(\tau(\theta)\,, \frac{\left((\tau(\theta)'\right)^2}{nI_X(\theta)}\right)$$

Then $\tau(\theta) = \theta$ and $\hat{\tau}(\theta) = \hat{\theta} = \frac{2}{\bar{X}}$ $\qquad \tau(\theta)' = 1$

$$f(x) = \theta^2 x e^{-x\theta}$$

$$lnf(x) = 2\ln(\theta) + \ln(x) - x\theta \ln e$$

$$lnf(x) = 2\ln(\theta) + \ln(x) - x$$

$$\frac{d}{d\theta} lnf(x) = \frac{2}{\theta} - x$$

$$\frac{d^2}{d\theta^2} lnf(x) = \frac{-2}{\theta^2}$$

or

$I(\theta) = E\left[\frac{d}{d\theta} lnf(x)\right]^2$	$I(\theta) = -E\left[\frac{d^2}{d\theta^2} lnf(x)\right]$
$= E\left[\frac{2}{\theta} - X\right]^2 = E\left[X - \frac{2}{\theta}\right]^2 = E[X - E(X)]^2$ $= V[X] = \frac{2}{\theta^2}$ $\Rightarrow I_X(\theta) = nI(\theta) = \frac{2n}{\theta^2}$	$= -E\left[\frac{-2}{\theta^2}\right] = -\left[\frac{-2}{\theta^2}\right] = \frac{2}{\theta^2}$ $\Rightarrow I_X(\theta) = nI(\theta) = \frac{2n}{\theta^2}$

$$\hat{\tau}(\theta) \to N\left(\tau(\theta)\,, \frac{\left((\tau(\theta)'\right)^2}{nI_X(\theta)}\right)$$

$$\frac{2}{\bar{X}} \to N\left(\theta, \frac{1}{\frac{2n}{\theta^2}}\right) \Rightarrow \frac{2}{\bar{X}} \to N\left(\theta, \frac{\theta^2}{2n}\right)$$

3.11: **Let $X_1, X_2, \ldots, X_n$ n>2, be a random sample from the binomial distribution $Binomail\,(1, \theta)$.**

$X_i \sim Binomial(1,\theta) \Rightarrow f(x) = \theta^x(1-\theta)^{1-x} \quad x = 0,1$

(a) Show that $T = X_1 + X_2 + \cdots + X_n = \sum_{i=1}^{n} x_i$ is a complete sufficient statistic for θ.

if $f(x) = a(\theta)\, b(x) e^{c(\theta)d(x)}$, then $T = \sum d(x_i)$ is complete minimal sufficient if p. 69

$f(x) = \theta^x(1-\theta)^{1-x}$ is a member of exponential family

$$f(x) = \theta^x(1-\theta)(1-\theta)^{-x}$$

$$= \frac{\theta^x}{(1-\theta)^x}(1-\theta)$$

$$= \left(\frac{\theta}{1-\theta}\right)^x(1-\theta)$$

$$= (1-\theta)e^{x\ln\left(\frac{\theta}{1-\theta}\right)}$$

$a(\theta) = (1-\theta)$, $\quad b(x) = 1$, $\quad c(\theta) = \ln\left(\frac{\theta}{1-\theta}\right)$, $\quad d(x) = x$

Then, $T = \sum_{i=1}^{n} d(x_i) = \sum_{i=1}^{n} x_i$ is a complete sufficient statistic for θ.

(b) Find the MVUE of θ.

$\boldsymbol{T = X_1 + X_2 + \cdots + X_n = \sum_{i=1}^{n} X_i} \sim Binomial(n,\theta) \Rightarrow E(T) = n\theta$

From (a), $T = \sum_{i=1}^{n} x_i$ is a complete sufficient statistic for θ

We have to find function of T which is unbiased estimator of θ, $E\big(\tau(T)\big) = \theta$

Since we know that $E(T) = n\theta \Rightarrow E\left(\frac{T}{n}\right) = \theta$

$$\Rightarrow \tau(T) = \frac{T}{n} \text{ is MVUE of } \theta$$

(c) **Let $T_2 = \frac{X_1+X_2}{2}$ and prove that T_2 is an unbiased estimator for θ.**

$$E(T_2) = E\left(\frac{X_1+X_2}{2}\right) = \frac{E(X_1)+E(X_2)}{2} = \frac{\theta+\theta}{2} = \theta.$$

(d) Find the approximate distribution of the MLE of 𝜃.

MLE:

(1) $L = \prod_{i=1}^{n} f(x_i) = \prod_{i=1}^{n} \theta.^{x_i}(1-\theta.).^{1-x_i} = \theta.^{\sum_{i=1}^{n} x_i}(1-\theta)^{n-\sum_{i=1}^{n} x_i}$

(2) $\log L = \sum_{i=1}^{n} x_i \log\theta. + (n - \sum_{i=1}^{n} x_i)\log(1-\theta)$

(3) $\frac{\partial}{\partial p}\log L = 0$ $\boxed{\frac{\partial \log L(\theta_i;x)}{\partial \theta_i} = 0}$ $p.48$

$$\Rightarrow \frac{\sum_{i=1}^{n} x_i}{\theta} - \frac{n-\sum_{i=1}^{n} x_i}{1-\theta} = 0$$

$$\Rightarrow \frac{(1-\theta)\sum_{i=1}^{n} x_i - \theta(n-\sum_{i=1}^{n} x_i)}{\theta(1-\theta)} = 0$$

$$\Rightarrow (1-\theta)\sum_{i=1}^{n} x_i - \theta(n-\sum_{i=1}^{n} x_i) = 0$$

$$\Rightarrow \sum_{i=1}^{n} x_i - \theta\sum_{i=1}^{n} x_i - n\theta + \theta\sum_{i=1}^{n} x_i = 0$$

$$\Rightarrow \sum_{i=1}^{n} x_i - n\theta = 0$$

$$\Rightarrow \theta = \frac{\sum_{i=1}^{n} x_i}{n}$$

$$\Rightarrow \hat{\theta} = \bar{X}$$

Suppose than $n \to \infty$ if $\hat{\tau}(\theta)$ be the MLE of $\tau(\theta)$, then $\hat{\tau}(\theta)$ has distribution as

$$\sqrt{n\left(\hat{\tau}(\theta) - \tau(\theta)\right)} \to N\left(0, \frac{\left((\tau(\theta)'\right)^2}{I_X(\theta)}\right) \quad or \quad \hat{\tau}(\theta) \to N\left(\tau(\theta), \frac{\left((\tau(\theta)'\right)^2}{nI_X(\theta)}\right)$$

Then $\tau(\theta) = \theta$ and $\hat{\tau}(\theta) = \hat{\theta} = \bar{X}$ $\quad \tau(\theta)' = 1$

$$f(x) = \theta^x(1-\theta)^{1-x}, \quad x = 0,1 \qquad E(X) = \theta \quad V(X) = \theta(1-\theta)$$

$$lnf(x) = ln\theta^x(1-\theta)^{1-x} = xln\theta + (1-x)\ln(1-\theta)$$

$$\frac{d}{d\theta}lnf(x) = \frac{x}{\theta} - \frac{1-x}{(1-\theta)}$$

$$\frac{d^2}{d\theta^2}lnf(x) = \frac{-x}{\theta^2} - \frac{(1-x)}{(1-\theta)^2}(-(-1))$$

$$= \frac{-x}{\theta^2} - \frac{(1-x)}{(1-\theta)^2}$$

or

$I(\theta) = E\left[\frac{d}{d\theta}lnf(x)\right]^2$	$I(\theta) = -E\left[\frac{d^2}{d\theta^2}lnf(x)\right]$
$= E\left[\frac{x}{\theta} - \frac{1-x}{(1-\theta)}\right]^2$ $= E\left[\frac{x(1-\theta)-\theta(1-x)}{\theta(1-\theta)}\right]^2$ $= E\left[\frac{x-x\theta-\theta+x\theta)}{\theta(1-\theta)}\right]^2$ $= E\left[\frac{x-\theta}{\theta(1-\theta)}\right]^2 = \frac{1}{\theta^2(1-\theta)^2}E[X-\theta]^2$ $= \frac{1}{\theta^2(1-\theta)^2}E[X-E(X)]^2$ $= \frac{1}{\theta^2(1-\theta)^2}V[X]$ $= \frac{1}{\theta^2(1-\theta)^2}\theta(1-\theta) = \frac{1}{\theta(1-\theta)}$ $\Rightarrow I_X(\theta) = nI(\theta) = \frac{n}{\theta(1-\theta)}$	$= -E\left[\frac{-X}{\theta^2} - \frac{(1-X)}{(1-\theta)^2}\right]$ $= -E\left[\frac{-X(1-\theta)^2-(1-X)\theta^2}{\theta^2(1-\theta)^2}\right]$ $= -E\left[\frac{2\theta X - X - \theta^2}{\theta^2(1-\theta)^2}\right] = \frac{-1}{\theta^2(1-\theta)^2}E[2\theta X - X - \theta^2]$ $= \frac{-1}{\theta^2(1-\theta)^2}[2\theta E(X) - E(X) - \theta^2]$ $= \frac{-1}{\theta^2(1-\theta)^2}[2\theta\theta - \theta - \theta^2]$ $= \frac{-1}{\theta^2(1-\theta)^2}[2\theta^2 - \theta - \theta^2]$ $= \frac{-1}{\theta^2(1-\theta)^2}[\theta^2 - \theta] = \frac{\theta-\theta^2}{\theta^2(1-\theta)^2} = \frac{\theta(1-\theta)}{\theta^2(1-\theta)^2}$ $= \frac{1}{\theta(1-\theta)}$ $\Rightarrow I_X(\theta) = nI(\theta) = \frac{n}{\theta(1-\theta)}$

$$\hat{\tau}(\theta) \to N\left(\tau(\theta), \frac{((\tau(\theta)')^2}{nI_X(\theta)}\right)$$

$$\bar{X} \to N\left(\theta, \frac{1}{\frac{n}{\theta(1-\theta)}}\right) \Rightarrow \bar{X} \to N\left(\theta, \frac{\theta(1-\theta)}{n}\right)$$

Chapter 4: Interval Estimation

4.1 Let the observed value of the mean ***X*** of a random sample of size 20 from a distribution that is ***N* (*μ*, 80)** be 81.2. Find a 95% confidence interval for ***μ***

Given $X_1, X_2, \dots, X_{20} \sim \text{Normal}(\mu, 80)$

$\sigma^2 = 80 \Rightarrow \sigma = \sqrt{80}$, $\bar{X} = 81.2$

since $X_i \sim \text{Normal}$ $\mu \in \left(\bar{X} \pm Z_{1-\frac{\alpha}{2}} \frac{\sigma}{\sqrt{n}}\right)$ if σ known *p*. 98

$\mu \in \left(\bar{X} \pm Z_{1-\frac{\alpha}{2}} \frac{\sigma}{\sqrt{n}}\right)$ $\mu \in (81.2 \pm 1.96 \frac{\sqrt{80}}{\sqrt{20}})$ $\mu \in (77.28, 85.12)$	$95\% \to \alpha = 0.05$ $\Rightarrow Z_{1-\frac{\alpha}{2}} = Z_{1-\frac{0.05}{2}}$ $= Z_{0.975} = 1.96$

4.2 Let ***X*** be the mean of a random sample of size ***n*** from a distribution that is ***N* (*μ*, 9).** Find ***n*** such that ***P*(*X̄* − 1 < *μ* < *X̄* + 1) = 0.90**, approximately.

$X_1, X_2, \dots, X_n \sim \text{Normal}(\mu, 9)$

$\sigma^2 = 9 \Rightarrow \sigma = 3$

since $X_i \sim \text{Normal}$ $\mu \in \left(\bar{X} \pm Z_{1-\frac{\alpha}{2}} \frac{\sigma}{\sqrt{n}}\right)$ if σ known *p*. 98

$P(\bar{X} - 1 < \mu < \bar{X} + 1) = 0.90$ $\bar{X} - Z_{0.95} \frac{\sigma}{\sqrt{n}} < \mu < \bar{X} + \boldsymbol{Z_{0.95} \frac{\sigma}{\sqrt{n}}}$ $\bar{X} - 1 < \mu < \bar{X} + \mathbf{1}$ $Z_{0.95} \frac{\sigma}{\sqrt{n}} = 1$ $1.645 \frac{3}{\sqrt{n}} = 1$ $n = 24.35 \approx 25$	$90\% \to \alpha = 0.1$ $\Rightarrow Z_{1-\frac{\alpha}{2}} = Z_{1-\frac{0.1}{2}}$ $= Z_{0.95} = 1.645$

4.3 Let a random sample of size 17 from the normal distribution $\boldsymbol{N(\mu,\sigma^2)}$ yield $\boldsymbol{\bar{X}= 4.7}$ and $\boldsymbol{S^2 = 5.76}$. Determine a 90% confidence interval for $\boldsymbol{\mu}$

Given $X_1, X_2, \dots, X_{17} \sim \text{Normal}(\mu, \sigma^2)$ $\bar{X} = 4.7$ $S^2 = 5.76$ n=17

since $X_i \sim \text{Normal}$ $\mu \in \left(\bar{X} \pm t_{1-\frac{\alpha}{2},n-1} \frac{s}{\sqrt{n}}\right)$ if σ unknown *p.* 98

$\mu \in \left(\bar{X} \pm t_{1-\frac{\alpha}{2},n-1} \frac{s}{\sqrt{n}}\right)$ $\mu \in \left(4.7 \pm 1.746 \frac{\sqrt{5.76}}{\sqrt{17}}\right)$ $\mu \in (3.6524\ , 5.7476)$	$90\% \to \alpha = 0.1$ $\Rightarrow t_{1-\frac{\alpha}{2},n-1} = t_{1-\frac{0.1}{2},17-1}$ $= t_{0.95,16} = 1.746$

4.4 If 8.6 7.9 8.3 8.4 6.4 8.4 9.8 7.2 7.8 7.5 are the observed values of a random sample of size 10 from a distribution that is $\boldsymbol{N(8,\sigma^2)}$, construct a 90% confidence interval for $\boldsymbol{\sigma^2}$.

since $X_i \sim \text{Normal}$ and μ known $\Rightarrow \frac{\sum_{i=1}^{n}(X_i-\mu)^2}{\chi^2_{\frac{\alpha}{2},n}} < \sigma^2 < \frac{\sum_{i=1}^{n}(X_i-\mu)^2}{\chi^2_{1-\frac{\alpha}{2},n}}$ p. 100

$\sum_{i=1}^{n}(X_i - 8)^2 = (8.6-8)^2 + (7.9-8)^2 + \dots + (7.5-8)^2 = 7.51$

$\frac{\sum_{i=1}^{n}(X_i-\mu)^2}{\chi^2_{\frac{\alpha}{2},n}} < \sigma^2 < \frac{\sum_{i=1}^{n}(X_i-\mu)^2}{\chi^2_{1-\frac{\alpha}{2},n}}$ $\frac{7.51}{18.31} < \sigma^2 < \frac{7.51}{3.94}$ $0.41 < \sigma^2 < 1.91$	$90\% \to \alpha = 0.1$ $\Rightarrow \chi^2_{\frac{\alpha}{2},n} = \chi^2_{\frac{0.1}{2},10} = \chi^2_{0.05,10} = 18.31$ $\Rightarrow \chi^2_{1-\frac{\alpha}{2},n} = \chi^2_{1-\frac{0.1}{2},10} = \chi^2_{0.95,10} = 3.94$

4.5 A random sample of size 15 from the normal distribution $\boldsymbol{N(\mu,\sigma^2)}$ yields $\boldsymbol{\bar{X} = 3.2}$ and $\boldsymbol{s^2 = 4.24}$. Determine a 90% confidence interval for $\boldsymbol{\sigma^2}$

since $X_i \sim \text{Normal}$ and μ unknown $\Rightarrow \frac{(n-1)s^2}{\chi^2_{\frac{\alpha}{2},n-1}} < \sigma^2 < \frac{(n-1)s^2}{\chi^2_{1-\frac{\alpha}{2},n-1}}$ p. 100

$\frac{(n-1)s^2}{\chi^2_{\frac{\alpha}{2},n-1}} < \sigma^2 < \frac{(n-1)s^2}{\chi^2_{1-\frac{\alpha}{2},n-1}}$ $\frac{(14)(4.24)}{23.68} < \sigma^2 < \frac{(14)(2.24)}{6.57}$ $2.51 < \sigma^2 < 9.035$	$90\% \to \alpha = 0.1$ $\Rightarrow \chi^2_{\frac{\alpha}{2},n-1} = \chi^2_{\frac{0.1}{2},15-1} = \chi^2_{0.05,14} = 23.68$ $\Rightarrow \chi^2_{1-\frac{\alpha}{2},n-1} = \chi^2_{1-\frac{0.1}{2},15-1} = \chi^2_{0.95,14} = 6.57$

4.6 Find a pivotal quantity based on a random sample of size n from $N(\theta, \theta^2)$ population, where $\theta > 0$. Use the pivotal quantity to set up a 1 - α confidence interval for θ.

Suppose that n<30 and since θ^2 unknown and since X ~Normal p.98 We can use the pivotal quantity $\frac{\bar{X}-\theta}{S/\sqrt{n}} \sim t_{(n-1)}$	Since X ~Normal and θ unknown p.100 We can use the pivotal quantity $\frac{(n-1)s^2}{\theta^2} \sim \chi^2_{n-1}$
$P\left(t_{\frac{\alpha}{2}} < \frac{\bar{X}-\theta}{S/\sqrt{n}} < t_{1-\frac{\alpha}{2}}\right) = 1 - \alpha$ $\Rightarrow t_{\frac{\alpha}{2},n-1} < \frac{\bar{X}-\theta}{S/\sqrt{n}} < t_{1-\frac{\alpha}{2},n-1}$ $\Rightarrow \frac{s}{\sqrt{n}} t_{\frac{\alpha}{2},n-1} < \bar{X} - \theta < \frac{s}{\sqrt{n}} t_{1-\frac{\alpha}{2},n-1}$ $\Rightarrow \bar{X} - \frac{s}{\sqrt{n}} t_{\frac{\alpha}{2},n-1} < \theta < \bar{X} + \frac{s}{\sqrt{n}} t_{1-\frac{\alpha}{2},n-1}$	$P\left(\chi^2_{1-\frac{\alpha}{2},n-1} < \frac{(n-1)s^2}{\theta^2} < \chi^2_{\frac{\alpha}{2},n-1}\right) = 1 - \alpha$ $\Rightarrow \frac{1}{\chi^2_{\frac{\alpha}{2},n-1}} < \frac{\theta^2}{(n-1)s^2} < \frac{1}{\chi^2_{1-\frac{\alpha}{2},n-1}}$ $\Rightarrow \frac{(n-1)s^2}{\chi^2_{\frac{\alpha}{2},n-1}} < \theta^2 < \frac{(n-1)s^2}{\chi^2_{1-\frac{\alpha}{2},n-1}}$ $\Rightarrow \sqrt{\frac{(n-1)s^2}{\chi^2_{\frac{\alpha}{2},n-1}}} < \theta < \sqrt{\frac{(n-1)s^2}{\chi^2_{1-\frac{\alpha}{2},n-1}}}$

لا تهمل باقي الاثباتات من صفحه 97 الى 101

4.7 Let $\bar{X}$ denote the mean of a random sample of size 25 from a gamma distribution with 4 and $\beta >$ **0.** Use the central limit theorem to find an approximate 95% confidence interval for β

Given that X~Gamma$(4\,,\beta\,) \Rightarrow \mu = E(X) = 4\beta \qquad V(X) = 4\beta^2$

By using the CLT $\Rightarrow$ $\bar{X}\sim\text{Normal}\left(\text{E(X)},\frac{V(X)}{n}\right) \Rightarrow \bar{X}\sim\text{Normal}\left(4\beta\,,\frac{4\beta^2}{n}\right)$

Then, $\frac{\bar{X}-4\beta}{\sqrt{4\beta^2/n}}\sim\text{Normal}(0,1)$

$$\frac{\bar{X}-4\beta}{\sqrt{4\beta^2/25}} = \frac{\bar{X}-4\beta}{2\beta/5}$$

$$= \frac{\bar{X}-4\beta}{2\beta/5}\left(\frac{5}{5}\right)$$

$$= \frac{5\bar{X}-20\beta}{2\beta}$$

$$= \frac{5\bar{X}}{2\beta} - \frac{20\beta}{2\beta}$$

$$= \frac{5\bar{X}}{2\beta} - 10 \quad \sim\text{Normal}(0,1)$$

$$P\left(Z_{\frac{\alpha}{2}} < \frac{5\bar{X}}{2\beta} - 10 < Z_{1-\frac{\alpha}{2}}\right) = 0.95 \qquad \boxed{Z_{\frac{\alpha}{2}} = -Z_{1-\frac{\alpha}{2}}}$$

$$\Rightarrow \quad Z_{\frac{\alpha}{2}} < \frac{5\bar{X}}{2\beta} - 10 < Z_{1-\frac{\alpha}{2}}$$

$$\Rightarrow -1.96 < \frac{5\bar{X}}{2\beta} - 10 < 1.96 \qquad \boxed{\begin{matrix} 1-\alpha = 0.95 \\ \alpha = 0.05 \end{matrix} \Rightarrow 1 - \frac{\alpha}{2} = 0.975 \Rightarrow Z_{1-\frac{\alpha}{2}} = Z_{0.975} = 1.96}$$

$$\Rightarrow \quad 8.04 < \frac{5\bar{X}}{2\beta} < 11.96$$

$$\Rightarrow \frac{1}{11.96} < \frac{2\beta}{5\bar{X}} < \frac{1}{8.04}$$

$$\Rightarrow \frac{5\bar{X}}{11.96} < 2\beta < \frac{5\bar{X}}{8.04}$$

$$\Rightarrow \frac{10\bar{X}}{11.96} < 4\beta < \frac{10\bar{X}}{8.04}$$

$$\mu = 4\beta \in \left(\frac{10\bar{X}}{11.96}\,,\frac{10\bar{X}}{8.04}\right)$$

4.8 **Let $X_1, X_2, \dots, X_6$** be a random sample of size 6 from a gamma distribution with parameters 1 and unknown $\boldsymbol{\beta} > \mathbf{0}$. Discuss the construction of a 98% confidence interval for $\boldsymbol{\beta}$

It is given that $X_i \sim \text{Gamma}(1, \beta)$ $\quad n = 6 \quad \Rightarrow M_{X_i}(t) = (1 - \beta t)^{-1}$

$$\Rightarrow \sum_{i=1}^{n} X_i \sim \text{Gamma}(n, \beta) \Rightarrow \sum_{i=1}^{6} X_i \sim \text{Gamma}(6, \beta)$$

$$\Rightarrow \frac{2}{\beta}\sum_{i=1}^{n} X_i \sim \text{Gamma}(n, 2) \Rightarrow \frac{2}{\beta}\sum_{i=1}^{6} X_i \sim \text{Gamma}(6,2)$$

We know that $\chi^2_v \equiv \text{Gamma}\left(\alpha = \frac{v}{2}, \beta = 2\right)$

Check by using the M.G.F

$$M_{\frac{2}{\beta}\sum_{i=1}^{n} X_i}(t) = E\left(e^{\sum_{i=1}^{n} X_i\left(\frac{2}{\beta}t\right)}\right)$$

$$= E\left(e^{X_1\left(\frac{2}{\beta}t\right)}\right) \times E\left(e^{X_2\left(\frac{2}{\beta}t\right)}\right) \times \dots \times E\left(e^{X_n\left(\frac{2}{\beta}t\right)}\right)$$

$$= \left(1 - \beta\frac{2}{\beta}t\right)^{-1} \times \left(1 - \beta\frac{2}{\beta}t\right)^{-1} \times \dots \times \left(1 - \beta\frac{2}{\beta}t\right)^{-1}$$

$$= (1 - 2t)^{-n}$$

We know that $\chi^2_v \equiv \text{Gamma}\left(\alpha = \frac{v}{2}, \beta = 2\right)$

We have $\frac{2}{\beta}\sum_{i=1}^{6} X_i \sim \text{Gamma}(6,2)$

$$\Rightarrow \frac{2}{\beta}\sum_{i=1}^{6} X_i \sim \chi^2_{v=12}$$

$\alpha = \frac{v}{2} \quad \Rightarrow 6 = \frac{v}{2} \quad \Rightarrow v = 12$

$$\chi^2_{1-\frac{\alpha}{2},12} < \frac{2}{\beta}\sum_{i=1}^{6} X_i < \chi^2_{\frac{\alpha}{2},12}$$

$$\frac{1}{\chi^2_{\frac{\alpha}{2},12}} < \frac{\beta}{2}\frac{1}{\sum_{i=1}^{6} X_i} < \frac{1}{\chi^2_{1-\frac{\alpha}{2},12}}$$

$$\frac{2}{\chi^2_{\frac{\alpha}{2},12}} < \beta\frac{1}{\sum_{i=1}^{6} X_i} < \frac{2}{\chi^2_{1-\frac{\alpha}{2},12}}$$

$$\frac{2\sum_{i=1}^{6} X_i}{\chi^2_{\frac{\alpha}{2},12}} < \beta < \frac{2\sum_{i=1}^{6} X_i}{\chi^2_{1-\frac{\alpha}{2},12}}$$

Chapter 5: Bayesian Estimation

$X\|\theta \sim f(x\|\theta)$	The distribution
$\Theta \sim h(\theta)$	Prior distribution.
$L(x\|\theta) = \prod_{i=1}^{n} f(x_i\|\theta)$	Joint conditional distribution of X given θ.
$g(x,\theta) = L(x\|\theta)h(\theta)$	Joint distribution of X and θ.
$g_1(x) = \begin{cases} \int_\theta g(x,\theta)d\theta & \text{if } \Theta \text{ is continuous} \\ \sum_\theta g(x,\theta) & \text{if } \Theta \text{ is discrete} \end{cases}$	The marginal distribution of X.
$k(\theta\|x) = \frac{g(x,\theta)}{g_1(x)} = \frac{L(x\|\theta)h(\theta)}{g_1(x)}$ $k(\theta\|x) \propto L(x\|\theta)h(\theta)$ $\boxed{k(\theta\|x) \text{ is proportional to } L(x\|\theta)h(\theta)}$	Posterior distribution. The conditional distribution of θ given the sample X.

5.1: Let $X_1, X_2, \dots, X_n$ be a random sample from Bernoulli with parameter θ, and the prior distribution of Θ is a uniform distribution, where $0 < \theta < 1$. Find the posterior distribution and the Bayes' point estimator of Θ when the loss function be the squared error loss function

Given that $X_i \sim Bernoulli(\theta) \Rightarrow f(x|\theta) = \theta^x(1-\theta)^{1-x}, \quad x = 0,1$

$$\theta \sim Uniform(0,1) \qquad \Rightarrow h(\theta) = 1\,, \quad 0 < \theta < 1$$

$$L(x|\theta) = \prod_{i=1}^{n} f(x_i|\theta) = \theta^{\sum_{i=1}^{n} x_i}(1-\theta)^{n-\sum_{i=1}^{n} x_i}$$

The posterior distribution:

$$k(\theta|x) \propto L(x|\theta)\underbrace{h(\theta)}_{1}$$

$\boxed{k(\theta|x) \propto L(x|\theta)h(\theta)}$ p. 112

$$k(\theta|x) \propto \theta^{\sum_{i=1}^{n} x_i+1-1}(1-\theta)^{n-\sum_{i=1}^{n} x_i+1-1}$$

Then $\theta|X \sim Beta\left(\sum_{i=1}^{n} x_i + 1\ ,\ n - \sum_{i=1}^{n} x_i + 1\right)$

$$\boxed{\begin{array}{c} X \sim Beta(a,b) \\ f(x) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} x^{a-1}[1-x]^{b-1} \\ E(X) = \frac{a}{a+b} \end{array}}$$

Then the Bayes Point estimator of θ is

$$E(\theta|X) = \frac{\sum_{i=1}^{n} x_i+1}{\sum_{i=1}^{n} x_i+1+n-\sum_{i=1}^{n} x_i+1} = \frac{\sum_{i=1}^{n} x_i+1}{n+2}$$

<u>5.2:</u> Let Y have a binominal distribution in which $n = 20$ and $p = \theta$. The prior probability on Θ is $Beta$ (a, b), where $a, b > 0$ are known constants. Find the following:

(a) Posterior distribution.

Given that $Y \sim Binomial(20, \theta) \Rightarrow f(y|\theta) = \binom{20}{y} \theta^{y}(1-\theta)^{20-y}, \quad y = 0,1,\dots,20$

$\theta \sim Beta(a, b)\,, a, b > 0 \qquad \Rightarrow h(\theta) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \theta^{a-1}[1-\theta]^{b-1}\ , \ 0 < \theta < 1$

$$L(y|\theta) = \prod_{i=1}^{20} f(y_i|\theta) = \prod_{i=1}^{20} \binom{20}{y_i} \theta^{y}(1-\theta)^{20-y}$$

$$= \left[\prod_{i=1}^{20} \binom{20}{y_i}\right] \theta^{\sum_{i=1}^{20} y_i}(1-\theta)^{400-\sum_{i=1}^{20} y_i}$$

$$K(\theta|y) \propto \theta^{\sum_{i=1}^{20} y_i}(1-\theta)^{400-\sum_{i=1}^{20} y_i}\ \theta^{a-1}[1-\theta]^{b-1}$$

$$\propto \theta^{\sum_{i=1}^{20} y_i}\theta^{a-1}\ (1-\theta)^{400-\sum_{i=1}^{20} y_i}\ [1-\theta]^{b-1}$$

$$\propto \theta^{\sum_{i=1}^{20} y_i + a - 1}(1-\theta)^{400-\sum_{i=1}^{20} y_i + b - 1}$$

$X \sim Beta(a, b)$
$f(x) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} x^{a-1}[1-x]^{b-1}$
$E(X) = \frac{a}{a+b}$

Then $\theta|Y \sim Beta\ (\sum_{i=1}^{20} y_i + a\,, 400 - \sum_{i=1}^{20} y_i + b)$

(b) Bayes' point estimate of Θ, when L $[\theta, \delta(y)] = [\theta - \delta(y)]^2$.

$$E(\theta|X\ ) = \frac{\sum_{i=1}^{20} y_i + a}{\sum_{i=1}^{20} y_i + a + 400 - \sum_{i=1}^{20} y_i + b} = \frac{\sum_{i=1}^{20} y_i + a}{400 + a + b}$$

5.3: Let $X_1, X_2, \dots, X_{10}$ denote a random sample from a Poisson distribution with mean $\theta; \theta > 0$. Let $Y = \sum_{i=1}^{10} X_i$. Use the loss function to be $L[\theta, \delta(y)] = [\theta - \delta(y)]^2$. If Θ has the pdf.

$h(\theta) = \frac{\theta^2 e^{-\frac{1}{2}\theta}}{16}$; $\theta > 0$. Find:

(a) The posterior distribution.

Given that $X_1, X_2, \dots, X_{10} \sim poisson(\theta), x = 0,1, \dots$

$\Rightarrow f(x|\theta) = \frac{e^{-\theta}\theta^x}{x!}$; $x = 0,1, \dots$

$h(\theta) = \frac{\theta^2 e^{-\frac{1}{2}\theta}}{16}$; $\theta > 0$

$$L(x|\theta) = \prod_{i=1}^{n} f(x_i|\theta) = \prod_{i=1}^{10} \frac{e^{-\theta}\theta^x}{x!} = \frac{e^{-10\theta} \ \theta^{\sum_{i=1}^{10} x_i}}{\prod_{i=1}^{10} x!}$$

$$k(\theta|x) \propto L(x|\theta)\ h(\theta)$$

$$\propto e^{-10\theta} \ \theta^{\sum_{i=1}^{n} x_i} \ \ \theta^2 e^{-\frac{1}{2}\theta}$$

$$\propto \ \theta^{\sum_{i=1}^{n} x_i} \ \theta^2 \ e^{-10\theta} \ \ e^{-\frac{1}{2}\theta}$$

$$\propto \ \theta^{\sum_{i=1}^{n} x_i + 2} \ \ e^{-10.5\theta}$$

$$\propto \ \theta^{\sum_{i=1}^{n} x_i + 2 + 1 - 1} \ \ e^{-10.5\theta}$$

$$\propto \ \theta^{\sum_{i=1}^{n} x_i + 3 - 1} \ \ e^{-10.5\theta}$$

$X \sim Gamma\left(n, \frac{1}{a}\right): \ f(x) = \frac{a^n}{\Gamma(n)} x^{n-1} e^{-ax}, E(\theta|X) = n \times \frac{1}{a}$

Then $\theta|X \sim Gamma\ (\sum_{i=1}^{n} x_i + 3, \frac{1}{10.5})$

(b) The Bayes' solution $\delta(y)$ for a point estimate for θ, when $Y = 22$.

$$E(\theta|X) = \frac{\sum_{i=1}^{10} x_i + 3}{10.5} = \frac{22+3}{10.5} = 2.38 \qquad (Y = \sum_{i=1}^{10} X_i = 22)$$

(C) 90% CI for θ $Y = 22$

We have from (a) $\theta|X \sim Gamma\,(\sum_{i=1}^{n} x_i + 3, \frac{1}{10.5})$

$X \sim Gamma(\alpha, 2) \equiv \chi_v^2 \quad \Rightarrow v = 2\alpha$

$\Rightarrow 2(10.5)\ \theta|X \sim Gamma\,(\sum_{i=1}^{n} x_i + 3, 2)$

$X \sim Gamma(\alpha, \theta)$
$\Rightarrow mX \sim Gamma(\alpha, m\theta)$

$\Rightarrow 2(10.5)\ \theta|X \sim \chi^2_{2(\sum_{i=1}^{n} x_i + 3)}$

$\Rightarrow 21\ \theta|X \sim \chi^2_{2(22+3)}$

$\Rightarrow 21\ \theta|X \sim \chi^2_{50}$

Then, $\chi^2_{1-\frac{\alpha}{2},50} < 21\theta < \chi^2_{\frac{\alpha}{2},50}$	$90\% \rightarrow \alpha = 0.1$
$34.76 < 21\theta < 67.5$	$\Rightarrow \chi^2_{\frac{0.1}{2},50} = \chi^2_{0.05,50} = 67.5$
$1.66 < \theta < 3.21$	$\Rightarrow \chi^2_{1-\frac{0.1}{2},50} = \chi^2_{0.95,50} = 34.76$

5.4: Let Y be the nth order statistic of a random sample of size n from a distribution with pdf $f(x) = \frac{1}{\theta}; 0 < x < \theta$, Take the loss function to be $L[\theta, \delta(y)] = [\theta - \delta(y)]^2$. Let θ be an observed value of the random variable Θ, which has pdf $h(\theta) = \frac{\beta\alpha^\beta}{\theta^{\beta+1}}; \ \alpha < \theta < \infty, \alpha > 0, \beta > 0$. Find the Bayes' solution $\delta(y)$ for a point estimate of θ.

Given that $f(x) = \frac{1}{\theta}, \quad 0 < x < \theta$

$$h(\theta) = \frac{\beta\alpha^\beta}{\theta^{\beta+1}}; \quad \alpha < \theta < \infty, \quad \alpha > 0, \quad \beta > 0$$

$$Y_n = max(X_1, X_2, \dots, X_n)$$

$$f_{Y_n}(y_n) = nf_X(y_n)[F_X(y_n)]^{n-1}; \ 0 \le y_n < \theta$$

$$= n\frac{1}{\theta}\left[\frac{y_n}{\theta}\right]^{n-1} = \frac{ny_n{}^{n-1}}{\theta^n}$$

$$K(\theta|y_n) \propto \frac{1}{\theta^n}\frac{1}{\theta^{\beta+1}}$$

We have proved in example (3.5) page:55

that $Y = \text{Maximum}(X_1, X_2, \dots, X_n)$ is a sufficient statistic

then $k(\theta|x) \propto f_{Y_n}(y_n|\theta)\, h(\theta)$ page 97

$$K(\theta|y_n) = c\frac{1}{\theta^{n+\beta+1}}; \ \alpha < \theta < \infty \qquad \text{p.111}$$

To find the constant c:

Since $K(\theta|y_n)$ is a pdf, then $\int_\alpha^\infty K(\theta|y_n)\, d\theta = 1$

$$\Rightarrow \int_\alpha^\infty c\frac{1}{\theta^{n+\beta+1}}\, d\theta = 1 \Rightarrow \int_\alpha^\infty c\theta^{-n-\beta-1}\, d\theta = 1$$

$$\Rightarrow c\left[\frac{\theta^{-n-\beta-1+1}}{-n-\beta-1+1}\right]_\alpha^\infty = 1 \quad \Rightarrow \frac{c}{-n-\beta}\left[\theta^{-n-\beta}\right]_\alpha^\infty = 1$$

$$\Rightarrow \frac{c}{-n-\beta}\left(0 - \alpha^{-n-\beta}\right) = 1$$

$$\Rightarrow \frac{c}{n+\beta}\alpha^{-(n+\beta)} = 1$$

$$\Rightarrow \frac{c}{\alpha^{(n+\beta)}n+\beta} = 1$$

$$\Rightarrow \boxed{C = \alpha^{(n+\beta)}n + \beta}$$

The Bayes point estimate of θ such that the loss function is the squared error loss is

$$E(\theta|Y_n) = \int_{\alpha}^{\infty} \theta K(\theta|y_n)\, d\theta$$

$$= \int_{\alpha}^{\infty} \theta \frac{(n+\beta)\alpha^{n+\beta}}{\theta^{n+\beta+1}}\, d\theta$$

$$= (n+\beta)\alpha^{n+\beta} \int_{\alpha}^{\infty} \theta^{-n-\beta}\, d\theta$$

$$= (n+\beta)\alpha^{n+\beta} \int_{\alpha}^{\infty} \theta^{-(n+\beta)}\, d\theta$$

$$= (n+\beta)\alpha^{n+\beta} \frac{\theta^{-(n+\beta)+1}}{-(n+\beta)+1}\Big|_{\alpha}^{\infty}$$

$$= \frac{n+\beta}{-(n+\beta)+1}\alpha^{n+\beta}(0 - \alpha^{-(n+\beta)+1})$$

$$= \frac{n+\beta}{-(n+\beta)+1}\alpha^{n+\beta}(-\alpha^{-(n+\beta)+1})$$

$$= \frac{n+\beta}{-(n+\beta)+1}(-\alpha^{n+\beta-(n+\beta)+1})$$

$$= \frac{n+\beta}{-(n+\beta)+1}(-\alpha)$$

$$= \frac{n+\beta}{-(n+\beta-1)}(-\alpha)$$

$$= \frac{n+\beta}{n+\beta-1}\alpha \quad \ldots. \quad (1)$$

5.5: In Exercise 5.4, let n=4 from the uniform pdf $f(x) = \frac{1}{\theta}; 0 < x < \theta$,and the prior θ pdf be $g(\theta) = \frac{2}{\theta^3}$; $1 < \theta < \infty$. Find:

(a) The Bayesian estimator $\delta(Y_4)$ of θ, based upon the sufficient statistic Y_4 , using the loss function $[\theta - \delta(Y_4)]^2$.

in 5.4 if $n = 4$ $f(x) = \frac{1}{\theta}$, $0 < x < \theta$

$$g(\theta) = \frac{2}{\theta^3},\ 1 < \theta < \infty \Rightarrow g(\theta) = \frac{\beta\alpha^\beta}{\theta^{\beta+1}} = \frac{2(1)^2}{\theta^{2+1}} = \frac{2}{\theta^3}$$

Y_4 is the maximum order statistic in the sample X_1, X_2, X_3, X_4

To find the Bayesian estimate of θ, we substitute in (1) by $n = 4, \beta = 2$ and $\alpha = 1$

$$E(\theta|Y_4) = \frac{n+\beta}{(n+\beta-1)}\alpha = \frac{4+2}{(4+2-1)}(1) = \frac{6}{5}$$

(b) The Bayesian estimator $\delta(Y_4)$ of θ, based upon the sufficient statistic Y_4, using the loss function $|\theta - \delta(Y_4)|$

It is the median m of $K(\theta|y_n)$; $\alpha < \theta < \infty$ which is the solution of $\int_\alpha^m K(\theta|y_n)d\theta = \frac{1}{2}$

$$\Rightarrow \int_\alpha^m \frac{(n+\beta)\alpha^{n+\beta}}{\theta^{n+\beta+1}} d\theta = \frac{1}{2} \quad (n = 4, \beta = 2 \text{ } and \text{ } \alpha = 1)$$

$$\Rightarrow \int_1^m \frac{6}{\theta^7} d\theta = \frac{1}{2}$$

$$\Rightarrow 6\int_1^m \theta^{-7} d\theta = \frac{1}{2}$$

$$\Rightarrow \frac{6}{-6}[\theta^{-6}]_1^m = \frac{1}{2}$$

$$\Rightarrow -(m^{-6} - 1) = \frac{1}{2}$$

$$\Rightarrow m^{-6} = \frac{1}{2}$$

$$\Rightarrow m^6 = 2 \Rightarrow \boxed{m = 1.1225}$$

5.6: $f(x|\theta) = \theta e^{-\theta x}$ $h(\theta) = \frac{\beta^\alpha}{\Gamma(\alpha)}\theta^{\alpha-1}e^{-\theta\beta}, \theta > 0$

(a) The posterior distribution of θ

$$L(x|\theta) = \prod_{i=1}^{n} f(x_i|\theta) = \prod_{i=1}^{n} \theta e^{-\theta x}$$

$$= \theta^n e^{-\theta\sum_{i=1}^{n} x_i}$$

$$K(\theta|X) \propto \theta^n e^{-\theta\sum_{i=1}^{n} x_i}\,\theta^{\alpha-1}e^{-\theta\beta}$$

$$\propto \theta^n \theta^{\alpha-1} e^{-\theta\sum_{i=1}^{n} x_i}\, e^{-\theta\beta}$$

$$\propto \theta^{n+\alpha-1}e^{-\theta(\beta+\sum_{i=1}^{n} x_i)}$$

$X \sim Gamma\left(n, \frac{1}{a}\right):\ f(x) = \frac{a^n}{\Gamma(n)}x^{n-1}e^{-ax}$
$E(\theta|X) = n \times \frac{1}{a}$

Then $\theta|X \sim Gamma\ (n+\alpha, \frac{1}{\beta+\sum_{i=1}^{n} x_i})$

(b) The Bayes Point estimate of θ use $L[\theta, \delta(y)] = [\theta - \delta(y)]^2$

$$E(\theta|X) = \frac{n+\alpha}{\beta+\sum_{i=1}^{n} x_i}$$

(c) If $X_1 = 2.5, X_2 = 3.61, X_3 = 4.8, X_4 = 2.74, X_5 = 3.95$ *and* $\alpha = 2$ $\beta = 4$

Calculate (b).

$$\sum_{i=1}^{5} x_i = 17.6,\ n = 5, \alpha = 2\ \ \beta = 4$$

$$E(\theta|X) = \frac{n+\alpha}{\beta+\sum_{i=1}^{n} x_i} = \frac{5+2}{4+17.6} = 0.3241$$

King Saud University

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STAT 340

Theory of Statistics 1

Exercises

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Chapter 1 Exercises: Introduction

1.1 Suppose that 4 out of 12 buildings in a certain city violate the building code. A building engineer randomly inspects a sample of 3 new buildings in the city.
   (a) Find the probability distribution function of the random variable X representing the number of buildings that violate the building code in the sample.
   (b) Find the probability that
      (i) none of the buildings in the sample violating the building code.
      (ii) one building in the sample violating the building code.
      (iii) at lease one building in the sample violating the building code.
   (c) Find the expected number of buildings in the sample that violate the building code.
   (d) Find Var(X).

1.2 On average, a certain intersection results in 3 traffic accidents per day. Assuming Poisson distribution,
   (i) what is the probability that at this intersection:
      (a) no accidents will occur in a given day?
      (b) More than 3 accidents will occur in a given day?
      (c) Exactly 5 accidents will occur in a period of two days?
   (ii) what is the average number of traffic accidents in a period of 4 days?

1.3 If the random variable X has a uniform distribution on the interval (0,10), then
   (a) $P(X < 6)$ equals to
   (b) The mean of X is
   (c) The variance X is

1.4 Suppose that Z is distributed according to the standard normal distribution. Then,
   (a) the area under the curve to the left of 1.43 is:
   (b) the area under the curve to the right of 0.89 is:
   (c) the area under the curve between 2.16 and 0.65 is:
   (d) the value of k such that $P(0.93< Z < k) = 0.0427$ is:

1.5 The finished inside diameter of a piston ring is normally distributed with a mean of 12 centimeters and a standard deviation of 0.03 centimeter. Find,
   (a) the proportion of rings that will have inside diameter less than 12.05 centimeters.
   (b) the proportion of rings that will have inside diameter exceeding 11.97 centimeters.
   (c) the probability that a piston ring will have an inside diameter between 11.95 and 12.05 centimeters.

1.6 Let X be $N(\mu, \sigma^2)$ so that $P(X < 89) = 0.90$ and $P(X < 94) = 0.95$. find μ and σ^2.

1.7 Assume the length (in minutes) of a particular type of a telephone conversation is a random variable with a probability density function of the form:

$$f(x) = \begin{cases} 0.2\, e^{-0.2x}; & x \geq 0 \\ 0; & \text{elsewhere} \end{cases}.$$

   Calculate:
   (a) $P(3 < x <10)$.
   (b) The cdf of X.
   (c) The mean and the variance of X.

1.8 Find the moment-generating function of X, if you know that $f(x) = 2e^{-2x}, x > 0$.

1.9 For a chi-squared distribution, find

(a) $\chi^2_{0.025}$ when $v = 15$.

(b) $\chi^2_{0.01}$ when $v = 7$.

(c) $\chi^2_{0.99}$ when $v = 22$.

1.10 If $(1-2t)^{-6}, t < \frac{1}{2}$, is the mgf of the random variable X, find $P(X < 5.23)$.

1.11 Find:

(a) $t_{0.95}$ when $v = 28$.

(b) $t_{0.005}$ when $v = 16$.

(c) $-t_{0.01}$ when $v = 4$.

(d) $P(T > 1.318)$ when $v = 24$.

(e) $P(-1.356 < T < 2.179)$ when $v = 12$.

1.12 If $f(x) = \theta x^{\theta-1}\ \ 0 < x < 1$, find the distribution of $Y = -lnX$.

1.13 If $f(x) = 1,\ 0 < x < 1$. Find the pdf of $Y = \sqrt{X}$.

1.14 If $X \sim Uniform(0,1)$, find the pdf of $Y = -2lnX$. Name the distribution and its parameter values.

1.15 Suppose independent random variables X and Y are such that $M_{X+Y}(t) = \frac{e^{2t}-1}{2t-t^2}$. If $f(x) = \lambda e^{-\lambda x}, x > 0$, what is the distribution of Y.

1.16 If $X_1 \sim \chi^2_n$ and $X_2 \sim \chi^2_m$ are independent random variables. Find the distribution of $Y = X_1 + X_2$.

Chapter 2 Exercises: Sampling Distribution

2.1 If e^{3t+4t^2} is the mgf of the random variable $\bar{X}$ with sample size 6, find $P(-2 < \bar{X} < 6)$.

2.2 Let $\bar{X}$ be the mean of a random sample of size 5 from a normal distribution with $\mu = 0$ and $\sigma^2 = 125$. Determine c so that $P(\bar{X} < c) = 0.975$.

2.3 Determine the mean and variance of the mean $\bar{X}$ of a random sample of size 9 from a distribution having pdf $f(x) = 4x^3, 0 < x < 1$, zero elsewhere.

2.4 Let $Z_1, Z_2, \dots\dots, Z_{16}$, be a random sample of size 16 from the standard normal distribution $N(0,1)$. Let $X_1, X_2, \dots., X_{64}$ be a random sample of size 64 from the normal distribution $N(\mu, 1)$. The two samples are independent.

(a) Find $P(Z_1 < 2)$.

(b) Find $P(\sum_{i=1}^{16} Z_i > 2)$

(c) Find $P\left(\sum_{i=1}^{16} {Z_i}^2 > 6.91\right)$

(d) Let S^2 be the sample variance of the first sample. Find c such that $P(S^2 > c) = 0.05$.

(e) What is the distribution of Y, where $Y = \sum_{i=1}^{16} Z_i^2 + \sum_{i=1}^{64} (X_i - \mu)^2$

(f) Find $E(Y)$.

(g) Find $Var(Y)$.

(h) Approximate $P(Y > 105)$.

(i) Find c such that $c\frac{\sum_{i=1}^{16} Z_1^2}{Y} \sim F_{16,80}$

(j) Let $Q \sim X_{60}^2$. Find c such that $P\left(\frac{Z_1}{\sqrt{Q}} < c\right) = 0.95$

(k) Find c such that $P\left(F_{60,20} > c\right) = 0.99$.

2.5 Let X be $N(5,10)$. Find $P(0.04 < (X-5)^2 < 38.4)$.

2.6 Let S^2 be the variance of a random sample of size 6 from the normal distribution $N(\mu, 12)$. Find

(a) Mean and variance of S^2.

(b) Distribution of S^2.

(c) $P(2.30 < S^2 < 22.2)$.

2.7 Let X_1, X_2 and X_3 be iid random variable, each with pdf $f(x) = e^{-x}, 0 < x < \infty$; and let $Y_1 < Y_2 < Y_3$ be the order statistics of the random variables. Find:

(a) The distribution of $Y_1 =$ minimum (X_1, X_2, X_3).

(b) $P(3 \le Y_1)$.

(c) The joint pdf of Y_2 and Y_3.

2.8 Let $Y_1 < Y_2 < \cdots < Y_n$ be the order statistics from a Weibull distribution. Find the distribution function and pdf of Y_1.

Chapter 3 Exercises: Point Estimation

3.1 Suppose $X_1, X_2, \dots, X_n$ is a random sample from gamma distribution:

$$f(x; \alpha, \beta) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x}, x > 0$$

Derive the MME for parameters α and β.

3.2 Find the MME and the MLE for the parameter p of Bernoulli distribution:

$$f(x;\ p) = p^x q^{1-x}, x = 0, 1.$$

Then, determine the unbiasedness, sufficiency and consistency of the MLE.

3.3 Let $f(x, \theta) = \theta e^{-\theta x}; x > 0,$ and let T be an estimator for $\tau(\theta)$. Study if T is unbiased, consistent estimator for $\tau(\theta)$, then compute MSE in the three cases:

(a) $T = \bar{X}$ and $\tau(\theta) = \frac{1}{\theta}$.

(b) $T = \frac{1}{\bar{X}}$ and $\tau(\theta) = \theta$.

(c) $T = \frac{n-1}{\sum X_i}$ and $\tau(\theta) = \theta$.

3.4 If $X_1, X_2, \dots, X_n$ be a random sample from $f(x; \theta)$. Show if the given statistic T is sufficient statistic for θ:

$$f(x; \theta) = e^{-(x-\theta)}, x > \theta\ ;\ \ T = Y_1 = \text{Minimum}(X_1, X_2, \dots, X_n).$$

3.5 Suppose for a given random variable T_1 and T_2 be two independents unbiased estimators for θ and with the same variance σ^2. Define two random variables as

$$Y = \frac{3T_1 + 2T_2}{5} \quad \text{and} \quad Z = \frac{T_1 + 2T_2}{3}$$

Find $MSE(Y)$ and $MSE(Z)$ and compare between them.

3.6 Let $f(x, \theta) = \frac{1}{\theta}; x \in (0, \theta)$, and let T be an estimator for θ. Study if T is unbiased, consistent and compute MSE, then compare between their variances for the following cases:

(a) $T = Y_1 = \text{Minimum}(X_1, X_2, \dots, X_n)$.

(b) $T = nY_1$.

(c) $T = 2\bar{X}$.

(d) $T = \frac{n+1}{n} Y_n$.

3.7 For a random sample $X_1, X_2, \dots, X_n$ drawn from the following distributions, find the Fisher information, $I_X(\theta)$:

(a) $Bernoulli(\theta)$.

(b) $Exponential\left(\frac{1}{\theta}\right)$

(c) $N(\theta, \sigma^2)$ when θ is unknown and σ^2 is known.

3.8 Let $X_1, X_2, \dots, X_n$ be a random sample drawn $N(\mu, \sigma^2), \sigma^2$ is known. Find:

(a) The CRLB for

i. $\tau(\mu) = \mu$ ii. $\tau(\mu) = e^\mu$ iii. $\tau(\mu) = \frac{1}{\mu+1}$

(b) The MVUE for μ.

3.10 Let $X_1, X_2, \dots, X_n$ be a random sample from a distribution with pdf

$$f(x;\theta) = \theta^2 x\, e^{-x\theta}, \qquad x > 0, \qquad \theta > 0$$

(a) Argue that $Y = \sum_{i=1}^{n} X_i$ is a complete sufficient statistic for θ.

(b) Compute $E\left(\frac{1}{Y}\right)$ and find the function of Y which is the unique MVUE of θ.

(c) Drive the MLE of θ and find the approximate distribution of it.

3.11 Let $X_1, X_2, \dots, X_n,\ n > 2,$ be a random sample from the binomial distribution $Binomial(1,\theta)$.

(a) Show that $T_1 = X_1 + X_2 + \dots + X_n$ is a complete sufficient statistic for θ.

(b) Find the MVUE of θ.

(c) Let $T_2 = \frac{X_1 + X_2}{2}$ and prove that T_2 is an unbiased estimator for θ.

(d) Find the approximate distribution of the MLE of θ.

Chapter 4 Exercises: Interval Estimation

4.1 Let the observed value of the mean $\bar{X}$ of a random sample of size 20 from a distribution that is $N(\mu, 80)$ be 81.2. Find a 95 percent confidence interval for μ.

4.2 Let $\bar{X}$ be the mean of a random sample of size *n* from a distribution that is $N(\mu, 9)$. Find *n* such that $P(\bar{X} - 1 < \mu < \bar{X} + 1) = 0.90$, approximately.

4.3 Let a random sample of size 17 from the normal distribution $N(\mu, \sigma^2)$ yield $\bar{x} = 4.7$ and $s^2 = 5.76$. Determine a 90% confidence interval for μ.

4.4 If 8.6, 7.9, 8.3, 8.4, 6.4, 8.4, 9.8, 7.2, 7.8, 7.5 are the observed values of a random sample of size 9 from a distribution that is $N(8, \sigma^2)$, construct a 90% confidence interval for σ^2.

4.5 A random sample of size 15 from the normal distribution $N(\mu, \sigma^2)$ yields $\bar{x} = 3.2$ and $s^2 = 4.24$. Determine a 90% confidence interval for σ^2.

4.6 Find a pivotal quantity based on a random sample of size *n* from $N(\theta, \theta^2)$ population, where $\theta > 0$. Use the pivotal quantity to set up a $1 - \alpha$ confidence interval for θ .

4.7 Let $\bar{X}$ denote the mean of a random sample of size 25 from a gamma distribution with 4 and $\beta > 0$. Use the central limit theorem to find an approximate 0.95 confidence interval for β.

4.8 Let $X_1, X_2, \dots, X_6$ be a random sample of size 6 from a gamma distribution with parameters 1 and unknown $\beta > 0$. Discuss the construction of a 98% confidence interval for β.

Chapter 5 Exercises: Bayesian Estimation

5.1 Let $X_1, X_2, \dots, X_n$ be a random sample from Bernoulli with parameter p, and the prior distribution of p is a uniform distribution, where $0 < p < 1$.

(a) Find the posterior distribution.

(b) Compute the Bayes' point estimator of p when the loss function be the squared error loss function.

(c) Construct 99% credible interval of p.

5.2 Let Y have a binominal distribution in which $n = 20$ and $p = \theta$. The prior probability on Θ is $Beta(a, b)$, where $a, b > 0$ are known constants. Find the following:

(a) Posterior distribution.

(b) Bayes' point estimate of Θ, when $\mathcal{L}[\theta, \delta(y)] = [\theta - \delta(y)]^2$.

(c) 90% credible interval of Θ.

5.3 Let $X_1, X_2, \dots, X_{10}$ denote a random sample from a Poisson distribution with mean $\theta, 0 < \theta < \infty$. Let $Y = \sum_i^{10} X_i$. Use the loss function to be $\mathcal{L}[\theta, \delta(y)] = [\theta - \delta(y)]^2$. If Θ has the pdf $h(\theta) = \frac{\theta^2 e^{-\frac{1}{2}\theta}}{16}$, for $0 < \theta < \infty$. Find:

(a) The posterior distribution.

(b) The Bayes' solution $\delta(y)$ for a point estimate for θ, when $Y = 22$.

(c) 95% credible interval of Θ.

5.4 Let Y_n be the nth order statistic of a random sample of size n from a distribution with pdf $f(x|\theta) = \frac{1}{\theta}, 0 < x \le \theta$, zero elsewhere. Take the loss function to be $\mathcal{L}[\theta, \delta(y)] = [\theta - \delta(y_n)]^2$. Let θ be an observed value of the random variable Θ, which has pdf $h(\theta) = \frac{\beta\alpha^\beta}{\theta^{\beta+1}}, \alpha < \theta < \infty$, zero elsewhere, with $\alpha > 0, \beta > 0$. Find the Bayes' solution $\delta(y_n)$ for a point estimate of θ and 90% credible interval of Θ.

5.5 In Exercise 5.5, let $n = 4$ from the uniform pdf $f(x, \theta) = \frac{1}{\theta}. 0 < x < \theta$, and the prior pdf be $g(\theta) = \frac{2}{\theta^3}, 1 < \theta < \infty$, zero elsewhere. Find:

(a) The Bayesian estimator $\delta(Y_4)$ of θ, based upon the sufficient statistic Y_4, using the loss function $[\theta - \delta(y_4)]^2$.

(b) The Bayesian estimator $\delta(Y_4)$ of θ, based upon the sufficient statistic Y_4, using the loss function $|\delta(y_4) - \theta|$.

(c) 98% credible interval of Θ.

5.6 Consider the model

$$X_i|\delta \sim \text{iid } Exponential\left(\frac{1}{\delta}\right)$$

$$\delta \sim Gamma\left(\alpha, \frac{1}{\beta}\right)$$

Find the following:

(a) Posterior distribution of δ.

(b) Bayes' point estimate of δ in different two ways.

(c) 95% credible interval of δ.

(d) If $X_1 = 2.5, X_2 = 3.61, X_3 = 4.8, X_4 = 2.74, X_5 = 3.95$ and $\alpha = 2, \beta = 4$. Calculate (b) and (c).

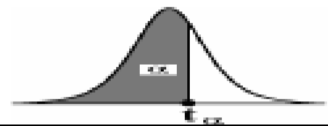
Standard Normal distribution $P(Z < z) = \alpha$

	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
-3.4	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0002
-3.3	0.0005	0.0005	0.0005	0.0004	0.0004	0.0004	0.0004	0.0004	0.0004	0.0003
-3.2	0.0007	0.0007	0.0006	0.0006	0.0006	0.0006	0.0006	0.0005	0.0005	0.0005
-3.1	0.0010	0.0009	0.0009	0.0009	0.0008	0.0008	0.0008	0.0008	0.0007	0.0007
-3.0	0.0013	0.0013	0.0013	0.0012	0.0012	0.0011	0.0011	0.0011	0.0010	0.0010
-2.9	0.0019	0.0018	0.0018	0.0017	0.0016	0.0016	0.0015	0.0015	0.0014	0.0014
-2.8	0.0026	0.0025	0.0024	0.0023	0.0023	0.0022	0.0021	0.0021	0.0020	0.0019
-2.7	0.0035	0.0034	0.0033	0.0032	0.0031	0.0030	0.0029	0.0028	0.0027	0.0026
-2.6	0.0047	0.0045	0.0044	0.0043	0.0041	0.0040	0.0039	0.0038	0.0037	0.0036
-2.5	0.0062	0.0060	0.0059	0.0057	0.0055	0.0054	0.0052	0.0051	0.0049	0.0048
-2.4	0.0082	0.0080	0.0078	0.0075	0.0073	0.0071	0.0069	0.0068	0.0066	0.0064
-2.3	0.0107	0.0104	0.0102	0.0099	0.0096	0.0094	0.0091	0.0089	0.0087	0.0084
-2.2	0.0139	0.0136	0.0132	0.0129	0.0125	0.0122	0.0119	0.0116	0.0113	0.0110
-2.1	0.0179	0.0174	0.0170	0.0166	0.0162	0.0158	0.0154	0.0150	0.0146	0.0143
-2.0	0.0228	0.0222	0.0217	0.0212	0.0207	0.0202	0.0197	0.0192	0.0188	0.0183
-1.9	0.0287	0.0281	0.0274	0.0268	0.0262	0.0256	0.0250	0.0244	0.0239	0.0233
-1.8	0.0359	0.0351	0.0344	0.0336	0.0329	0.0322	0.0314	0.0307	0.0301	0.0294
-1.7	0.0446	0.0436	0.0427	0.0418	0.0409	0.0401	0.0392	0.0384	0.0375	0.0367
-1.6	0.0548	0.0537	0.0526	0.0516	0.0505	0.0495	0.0485	0.0475	0.0465	0.0455
-1.5	0.0668	0.0655	0.0643	0.0630	0.0618	0.0606	0.0594	0.0582	0.0571	0.0559
-1.4	0.0808	0.0793	0.0778	0.0764	0.0749	0.0735	0.0721	0.0708	0.0694	0.0681
-1.3	0.0968	0.0951	0.0934	0.0918	0.0901	0.0885	0.0869	0.0853	0.0838	0.0823
-1.2	0.1151	0.1131	0.1112	0.1093	0.1075	0.1056	0.1038	0.1020	0.1003	0.0985
-1.1	0.1357	0.1335	0.1314	0.1292	0.1271	0.1251	0.1230	0.1210	0.1190	0.1170
-1.0	0.1587	0.1562	0.1539	0.1515	0.1492	0.1469	0.1446	0.1423	0.1401	0.1379
-0.9	0.1841	0.1814	0.1788	0.1762	0.1736	0.1711	0.1685	0.1660	0.1635	0.1611
-0.8	0.2119	0.2090	0.2061	0.2033	0.2005	0.1977	0.1949	0.1922	0.1894	0.1867
-0.7	0.2420	0.2389	0.2358	0.2327	0.2296	0.2266	0.2236	0.2206	0.2177	0.2148
-0.6	0.2743	0.2709	0.2676	0.2643	0.2611	0.2578	0.2546	0.2514	0.2483	0.2451
-0.5	0.3085	0.3050	0.3015	0.2981	0.2946	0.2912	0.2877	0.2843	0.2810	0.2776
-0.4	0.3446	0.3409	0.3372	0.3336	0.3300	0.3264	0.3228	0.3192	0.3156	0.3121
-0.3	0.3821	0.3783	0.3745	0.3707	0.3669	0.3632	0.3594	0.3557	0.3520	0.3483
-0.2	0.4207	0.4168	0.4129	0.4090	0.4052	0.4013	0.3974	0.3936	0.3897	0.3859
-0.1	0.4602	0.4562	0.4522	0.4483	0.4443	0.4404	0.4364	0.4325	0.4286	0.4247
-0.0	0.5000	0.4960	0.4920	0.4880	0.4840	0.4801	0.4761	0.4721	0.4681	0.4641
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998

Chi-square distribution $\chi^2_{v,\alpha}$: $P\left(\chi^2 > \chi^2_{v,\alpha}\right) = \alpha$

v	α														
	0.001	**0.005**	**0.010**	**0.025**	**0.050**	**0.100**	**0.250**	**0.500**	**0.750**	**0.900**	**0.950**	**0.975**	**0.990**	**0.995**	**0.999**
1	10.83	7.88	6.63	5.02	3.84	2.71	1.32	0.45	0.10	0.02	0.00	0.00	0.00	0.00	0.00
2	13.82	10.60	9.21	7.38	5.99	4.61	2.77	1.39	0.58	0.21	0.10	0.05	0.02	0.01	0.00
3	16.27	12.84	11.34	9.35	7.81	6.25	4.11	2.37	1.21	0.58	0.35	0.22	0.11	0.07	0.02
4	18.47	14.86	13.28	11.14	9.49	7.78	5.39	3.36	1.92	1.06	0.71	0.48	0.30	0.21	0.09
5	20.52	16.75	15.09	12.83	11.07	9.24	6.63	4.35	2.67	1.61	1.15	0.83	0.55	0.41	0.21
6	22.46	18.55	16.81	14.45	12.59	10.64	7.84	5.35	3.45	2.20	1.64	1.24	0.87	0.68	0.38
7	24.32	20.28	18.48	16.01	14.07	12.02	9.04	6.35	4.25	2.83	2.17	1.69	1.24	0.99	0.60
8	26.12	21.95	20.09	17.53	15.51	13.36	10.22	7.34	5.07	3.49	2.73	2.18	1.65	1.34	0.86
9	27.88	23.59	21.67	19.02	16.92	14.68	11.39	8.34	5.90	4.17	3.33	2.70	2.09	1.73	1.15
10	29.59	25.19	23.21	20.48	18.31	15.99	12.55	9.34	6.74	4.87	3.94	3.25	2.56	2.16	1.48
11	31.26	26.76	24.72	21.92	19.68	17.28	13.70	10.34	7.58	5.58	4.57	3.82	3.05	2.60	1.83
12	32.91	28.30	26.22	23.34	21.03	18.55	14.85	11.34	8.44	6.30	5.23	4.40	3.57	3.07	2.21
13	34.53	29.82	27.69	24.74	22.36	19.81	15.98	12.34	9.30	7.04	5.89	5.01	4.11	3.57	2.62
14	36.12	31.32	29.14	26.12	23.68	21.06	17.12	13.34	10.17	7.79	6.57	5.63	4.66	4.07	3.04
15	37.70	32.80	30.58	27.49	25.00	22.31	18.25	14.34	11.04	8.55	7.26	6.26	5.23	4.60	3.48
16	39.25	34.27	32.00	28.85	26.30	23.54	19.37	15.34	11.91	9.31	7.96	6.91	5.81	5.14	3.94
17	40.79	35.72	33.41	30.19	27.59	24.77	20.49	16.34	12.79	10.09	8.67	7.56	6.41	5.70	4.42
18	42.31	37.16	34.81	31.53	28.87	25.99	21.60	17.34	13.68	10.86	9.39	8.23	7.01	6.26	4.90
19	43.82	38.58	36.19	32.85	30.14	27.20	22.72	18.34	14.56	11.65	10.12	8.91	7.63	6.84	5.41
20	45.31	40.00	37.57	34.17	31.41	28.41	23.83	19.34	15.45	12.44	10.85	9.59	8.26	7.43	5.92
21	46.80	41.40	38.93	35.48	32.67	29.62	24.93	20.34	16.34	13.24	11.59	10.28	8.90	8.03	6.45
22	48.27	42.80	40.29	36.78	33.92	30.81	26.04	21.34	17.24	14.04	12.34	10.98	9.54	8.64	6.98
23	49.73	44.18	41.64	38.08	35.17	32.01	27.14	22.34	18.14	14.85	13.09	11.69	10.20	9.26	7.53
24	51.18	45.56	42.98	39.36	36.42	33.20	28.24	23.34	19.04	15.66	13.85	12.40	10.86	9.89	8.08
25	52.62	46.93	44.31	40.65	37.65	34.38	29.34	24.34	19.94	16.47	14.61	13.12	11.52	10.52	8.65
30	59.70	53.67	50.89	46.98	43.77	40.26	34.80	29.34	24.48	20.60	18.49	16.79	14.95	13.79	11.59
35	66.62	60.27	57.34	53.20	49.80	46.06	40.22	34.34	29.05	24.80	22.47	20.57	18.51	17.19	14.69
40	73.40	66.77	63.69	59.34	55.76	51.81	45.62	39.34	33.66	29.05	26.51	24.43	22.16	20.71	17.92
45	80.08	73.17	69.96	65.41	61.66	57.51	50.98	44.34	38.29	33.35	30.61	28.37	25.90	24.31	21.25
50	86.66	79.49	76.15	71.42	67.50	63.17	56.33	49.33	42.94	37.69	34.76	32.36	29.71	27.99	24.67
60	99.61	91.95	88.38	83.30	79.08	74.40	66.98	59.33	52.29	46.46	43.19	40.48	37.48	35.53	31.74
70	112.32	104.21	100.43	95.02	90.53	85.53	77.58	69.33	61.70	55.33	51.74	48.76	45.44	43.28	39.04
80	124.84	116.32	112.33	106.63	101.88	96.58	88.13	79.33	71.14	64.28	60.39	57.15	53.54	51.17	46.52
90	137.21	128.30	124.12	118.14	113.15	107.57	98.65	89.33	80.62	73.29	69.13	65.65	61.75	59.20	54.16
100	149.45	140.17	135.81	129.56	124.34	118.50	109.14	99.33	90.13	82.36	77.93	74.22	70.06	67.33	61.92

Critical Values of the t-distribution (t_α)



ν=df	$t_{0.90}$	$t_{0.95}$	$t_{0.975}$	$t_{0.99}$	$t_{0.995}$
1	3.078	6.314	12.706	31.821	63.657
2	1.886	2.920	4.303	6.965	9.925
3	1.638	2.353	3.182	4.541	5.841
4	1.533	2.132	2.776	3.747	4.604
5	1.476	2.015	2.571	3.365	4.032
6	1.440	1.943	2.447	3.143	3.707
7	1.415	1.895	2.365	2.998	3.499
8	1.397	1.860	2.306	2.896	3.355
9	1.383	1.833	2.262	2.821	3.250
10	1.372	1.812	2.228	2.764	3.169
11	1.363	1.796	2.201	2.718	3.106
12	1.356	1.782	2.179	2.681	3.055
13	1.350	1.771	2.160	2.650	3.012
14	1.345	1.761	2.145	2.624	2.977
15	1.341	1.753	2.131	2.602	2.947
16	1.337	1.746	2.120	2.583	2.921
17	1.333	1.740	2.110	2.567	2.898
18	1.330	1.734	2.101	2.552	2.878
19	1.328	1.729	2.093	2.539	2.861
20	1.325	1.725	2.086	2.528	2.845
21	1.323	1.721	2.080	2.518	2.831
22	1.321	1.717	2.074	2.508	2.819
23	1.319	1.714	2.069	2.500	2.807
24	1.318	1.711	2.064	2.492	2.797
25	1.316	1.708	2.060	2.485	2.787
26	1.315	1.706	2.056	2.479	2.779
27	1.314	1.703	2.052	2.473	2.771
28	1.313	1.701	2.048	2.467	2.763
29	1.311	1.699	2.045	2.462	2.756
30	1.310	1.697	2.042	2.457	2.750
35	1.3062	1.6896	2.0301	2.4377	2.7238
40	1.3030	1.6840	2.0210	2.4230	2.7040
45	1.3006	1.6794	2.0141	2.4121	2.6896
50	1.2987	1.6759	2.0086	2.4033	2.6778
60	1.2958	1.6706	2.0003	2.3901	2.6603
70	1.2938	1.6669	1.9944	2.3808	2.6479
80	1.2922	1.6641	1.9901	2.3739	2.6387
90	1.2910	1.6620	1.9867	2.3685	2.6316
100	1.2901	1.6602	1.9840	2.3642	2.6259
120	1.2886	1.6577	1.9799	2.3578	2.6174
140	1.2876	1.6558	1.9771	2.3533	2.6114
160	1.2869	1.6544	1.9749	2.3499	2.6069
180	1.2863	1.6534	1.9732	2.3472	2.6034
200	1.2858	1.6525	1.9719	2.3451	2.6006
∞	1.282	1.645	1.960	2.326	2.576