King Saud University

College of Science

Department of Statistics and Operations Research

STAT 340

Theory of Statistics 1

Exercises

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Chapter 1 Exercises: Introduction

1.1 Suppose that 4 out of 12 buildings in a certain city violate the building code. A building engineer randomly inspects a sample of 3 new buildings in the city.

(a) Find the probability distribution function of the random variable X representing the number of buildings that violate the building code in the sample.

(b) Find the probability that

(i) none of the buildings in the sample violating the building code.

(ii) one building in the sample violating the building code.

(iii) at lease one building in the sample violating the building code.

(c) Find the expected number of buildings in the sample that violate the building code.

(d) Find Var(X).

1.2 On average, a certain intersection results in 3 traffic accidents per day. Assuming Poisson distribution,

(i) what is the probability that at this intersection:

(a) no accidents will occur in a given day?

(b) More than 3 accidents will occur in a given day?

(c) Exactly 5 accidents will occur in a period of two days?

(ii) what is the average number of traffic accidents in a period of 4 days?

1.3 If the random variable X has a uniform distribution on the interval (0,10), then

(a) $P(X < 6)$ equals to

(b) The mean of X is

(c) The variance X is

1.4 Suppose that Z is distributed according to the standard normal distribution. Then,

(a) the area under the curve to the left of 1.43 is:

(b) the area under the curve to the right of 0.89 is:

(c) the area under the curve between 2.16 and 0.65 is:

(d) the value of k such that $P(0.93< Z < k) = 0.0427$ is:

1.5 The finished inside diameter of a piston ring is normally distributed with a mean of 12 centimeters and a standard deviation of 0.03 centimeter. Find,

(a) the proportion of rings that will have inside diameter less than 12.05 centimeters.

(b) the proportion of rings that will have inside diameter exceeding 11.97 centimeters.

(c) the probability that a piston ring will have an inside diameter between 11.95 and 12.05 centimeters.

1.6 Let X be $N(\mu, \sigma^2)$ so that $P(X < 89) = 0.90$ and $P(X < 94) = 0.95$. find μ and σ^2.

1.7 Assume the length (in minutes) of a particular type of a telephone conversation is a random variable with a probability density function of the form:

$$f(x) = \begin{cases} 0.2\, e^{-0.2x}; & x \geq 0 \\ 0; & \text{elsewhere} \end{cases}.$$

Calculate:

(a) $P(3 < x < 10)$.

(b) The cdf of X.

(c) The mean and the variance of X.

1.8 Find the moment-generating function of X, if you know that $f(x) = 2e^{-2x}, x > 0$.

1.9 For a chi-squared distribution, find

(a) $\chi^2_{0.025}$ when $v = 15$.

(b) $\chi^2_{0.01}$ when $v = 7$.

(c) $\chi^2_{0.99}$ when $v = 22$.

1.10 If $(1-2t)^{-6}, t < \frac{1}{2}$, is the mgf of the random variable X, find $P(X < 5.23)$.

1.11 Find:

(a) $t_{0.95}$ when $v = 28$.

(b) $t_{0.005}$ when $v = 16$.

(c) $-t_{0.01}$ when $v = 4$.

(d) $P(T > 1.318)$ when $v = 24$.

(e) $P(-1.356 < T < 2.179)$ when $v = 12$.

1.12 If $f(x) = \theta x^{\theta-1}$ $0 < x < 1$, find the distribution of $Y = -lnX$.

1.13 If $f(x) = 1,\ 0 < x < 1$. Find the pdf of $Y = \sqrt{X}$.

1.14 If $X \sim Uniform(0,1)$, find the pdf of $Y = -2lnX$. Name the distribution and its parameter values.

1.15 Suppose independent random variables X and Y are such that $M_{X+Y}(t) = \frac{e^{2t}-1}{2t-t^2}$. If $f(x) = \lambda e^{-\lambda x}, x > 0$, what is the distribution of Y.

1.16 If $X_1 \sim \chi^2_n$ and $X_2 \sim \chi^2_m$ are independent random variables. Find the distribution of $Y = X_1 + X_2$.

Chapter 2 Exercises: Sampling Distribution

2.1 If e^{3t+4t^2} is the mgf of the random variable $\bar{X}$ with sample size 6, find $P(-2 < \bar{X} < 6)$.

2.2 Let $\bar{X}$ be the mean of a random sample of size 5 from a normal distribution with $\mu = 0$ and $\sigma^2 = 125$. Determine c so that $P(\bar{X} < c) = 0.975$.

2.3 Determine the mean and variance of the mean $\bar{X}$ of a random sample of size 9 from a distribution having pdf $f(x) = 4x^3, 0 < x < 1$, zero elsewhere.

2.4 Let $Z_1, Z_2, \dots \dots, Z_{16}$, be a random sample of size 16 from the standard normal distribution $N(0,1)$. Let $X_1, X_2, \dots., X_{64}$ be a random sample of size 64 from the normal distribution $N(\mu, 1)$. The two samples are independent.

(a) Find $P(Z_1 < 2)$.

(b) Find $P(\sum_{i=1}^{16} Z_i > 2)$

(c) Find $P\left(\sum_{i=1}^{16} {Z_i}^2 > 6.91\right)$

(d) Let S^2 be the sample variance of the first sample. Find c such that $P(S^2 > c) = 0.05$.

(e) What is the distribution of Y, where $Y = \sum_{i=1}^{16} Z_i^2 + \sum_{i=1}^{64} (X_i - \mu)^2$

(f) Find $E(Y)$.

(g) Find $Var(Y)$.

(h) Approximate $P(Y > 105)$.

(i) Find c such that $c\frac{\sum_{i=1}^{16} Z_1^2}{Y} \sim F_{16,80}$

(j) Let $Q \sim X_{60}^2$. Find c such that $P\left(\frac{Z_1}{\sqrt{Q}} < c\right) = 0.95$

(k) Find c such that $P\left(F_{60,20} > c\right) = 0.99$.

2.5 Let X be $N(5,10)$. Find $P(0.04 < (X-5)^2 < 38.4)$.

2.6 Let S^2 be the variance of a random sample of size 6 from the normal distribution $N(\mu, 12)$. Find

(a) Mean and variance of S^2.

(b) Distribution of S^2.

(c) $P(2.30 < S^2 < 22.2)$.

2.7 Let X_1, X_2 and X_3 be iid random variable, each with pdf $f(x) = e^{-x}, 0 < x < \infty$; and let $Y_1 < Y_2 < Y_3$ be the order statistics of the random variables. Find:

(a) The distribution of $Y_1 =$ minimum (X_1, X_2, X_3).

(b) $P(3 \leq Y_1)$.

(c) The joint pdf of Y_2 and Y_3.

2.8 Let $Y_1 < Y_2 < \cdots < Y_n$ be the order statistics from a Weibull distribution. Find the distribution function and pdf of Y_1.

Chapter 3 Exercises: Point Estimation

3.1 Suppose $X_1, X_2, \dots, X_n$ is a random sample from gamma distribution:

$$f(x;\alpha,\beta) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x}, x > 0$$

Derive the MME for parameters α and β.

3.2 Find the MME and the MLE for the parameter p of Bernoulli distribution:

$$f(x;\, p) = p^x q^{1-x}, x = 0, 1.$$

Then, determine the unbiasedness, sufficiency and consistency of the MLE.

3.3 Let $f(x,\theta) = \theta e^{-\theta x}; x > 0,$ and let T be an estimator for $\tau(\theta)$. Study if T is unbiased, consistent estimator for $\tau(\theta)$, then compute MSE in the three cases:

(a) $T = \bar{X}$ and $\tau(\theta) = \frac{1}{\theta}$.

(b) $T = \frac{1}{\bar{X}}$ and $\tau(\theta) = \theta$.

(c) $T = \frac{n-1}{\sum X_i}$ and $\tau(\theta) = \theta$.

3.4 If $X_1, X_2, \dots, X_n$ be a random sample from $f(x;\theta)$. Show if the given statistic T is sufficient statistic for θ:

$$f(x;\theta) = e^{-(x-\theta)}, x > \theta\,;\ \ T = Y_1 = \text{Minimum}(X_1, X_2, \dots, X_n).$$

3.5 Suppose for a given random variable T_1 and T_2 be two independents unbiased estimators for θ and with the same variance σ^2. Define two random variables as

$$Y = \frac{3T_1 + 2T_2}{5} \quad \text{and} \quad Z = \frac{T_1 + 2T_2}{3}$$

Find $MSE(Y)$ and $MSE(Z)$ and compare between them.

3.6 Let $f(x,\theta) = \frac{1}{\theta}; x \in (0,\theta)$, and let T be an estimator for θ. Study if T is unbiased, consistent and compute MSE, then compare between their variances for the following cases:

(a) $T = Y_1 = \text{Minimum}(X_1, X_2, \dots, X_n)$.

(b) $T = nY_1$.

(c) $T = 2\bar{X}$.

(d) $T = \frac{n+1}{n} Y_n$.

3.7 For a random sample $X_1, X_2, \dots, X_n$ drawn from the following distributions, find the Fisher information, $I_X(\theta)$:

(a) $Bernoulli(\theta)$.

(b) $Exponential\left(\frac{1}{\theta}\right)$

(c) $N(\theta, \sigma^2)$ when θ is unknown and σ^2 is known.

3.8 Let $X_1, X_2, \dots, X_n$ be a random sample drawn $N(\mu, \sigma^2), \sigma^2$ is known. Find:

(a) The CRLB for

i. $\tau(\mu) = \mu$ ii. $\tau(\mu) = e^\mu$ iii. $\tau(\mu) = \frac{1}{\mu+1}$

(b) The MVUE for μ.

3.10 Let $X_1, X_2, \dots, X_n$ be a random sample from a distribution with pdf

$$f(x;\theta) = \theta^2 x\, e^{-x\theta}, \qquad x > 0, \qquad \theta > 0$$

(a) Argue that $Y = \sum_{i=1}^{n} X_i$ is a complete sufficient statistic for θ.

(b) Compute $E\left(\frac{1}{Y}\right)$ and find the function of Y which is the unique MVUE of θ.

(c) Drive the MLE of θ and find the approximate distribution of it.

3.11 Let $X_1, X_2, \dots, X_n$, $n > 2$, be a random sample from the binomial distribution $Binomial(1, \theta)$.

(a) Show that $T_1 = X_1 + X_2 + \dots + X_n$ is a complete sufficient statistic for θ.

(b) Find the MVUE of θ.

(c) Let $T_2 = \frac{X_1 + X_2}{2}$ and prove that T_2 is an unbiased estimator for θ.

(d) Find the approximate distribution of the MLE of θ.

Chapter 4 Exercises: Interval Estimation

4.1 Let the observed value of the mean $\bar{X}$ of a random sample of size 20 from a distribution that is $N(\mu, 80)$ be 81.2. Find a 95 percent confidence interval for μ.

4.2 Let $\bar{X}$ be the mean of a random sample of size *n* from a distribution that is $N(\mu, 9)$. Find *n* such that $P(\bar{X} - 1 < \mu < \bar{X} + 1) = 0.90$, approximately.

4.3 Let a random sample of size 17 from the normal distribution $N(\mu, \sigma^2)$ yield $\bar{x} = 4.7$ and $s^2 = 5.76$. Determine a 90% confidence interval for μ.

4.4 If 8.6, 7.9, 8.3, 8.4, 6.4, 8.4, 9.8, 7.2, 7.8, 7.5 are the observed values of a random sample of size 9 from a distribution that is $N(8, \sigma^2)$, construct a 90% confidence interval for σ^2.

4.5 A random sample of size 15 from the normal distribution $N(\mu, \sigma^2)$ yields $\bar{x} = 3.2$ and $s^2 = 4.24$. Determine a 90% confidence interval for σ^2.

4.6 Find a pivotal quantity based on a random sample of size *n* from $N(\theta, \theta^2)$ population, where $\theta > 0$. Use the pivotal quantity to set up a $1 - \alpha$ confidence interval for θ .

4.7 Let $\bar{X}$ denote the mean of a random sample of size 25 from a gamma distribution with 4 and $\beta > 0$. Use the central limit theorem to find an approximate 0.95 confidence interval for β.

4.8 Let $X_1, X_2, \dots, X_6$ be a random sample of size 6 from a gamma distribution with parameters 1 and unknown $\beta > 0$. Discuss the construction of a 98% confidence interval for β.

Chapter 5 Exercises: Bayesian Estimation

5.1 Let $X_1, X_2, \dots, X_n$ be a random sample from Bernoulli with parameter p, and the prior distribution of p is a uniform distribution, where $0 < p < 1$.
(a) Find the posterior distribution.
(b) Compute the Bayes' point estimator of p when the loss function be the squared error loss function.
(c) Construct 99% credible interval of p.

5.2 Let Y have a binominal distribution in which $n = 20$ and $p = \theta$. The prior probability on Θ is $Beta(a, b)$, where $a, b > 0$ are known constants. Find the following:
(a) Posterior distribution.
(b) Bayes' point estimate of Θ, when $\mathcal{L}[\theta, \delta(y)] = [\theta - \delta(y)]^2$.
(c) 90% credible interval of Θ.

5.3 Let $X_1, X_2, \dots, X_{10}$ denote a random sample from a Poisson distribution with mean $\theta, 0 < \theta < \infty$. Let $Y = \sum_i^{10} X_i$. Use the loss function to be $\mathcal{L}[\theta, \delta(y)] = [\theta - \delta(y)]^2$. If Θ has the pdf $h(\theta) = \frac{\theta^2 e^{-\frac{1}{2}\theta}}{16}$, for $0 < \theta < \infty$. Find:
(a) The posterior distribution.
(b) The Bayes' solution $\delta(y)$ for a point estimate for θ, when $Y = 22$.
(c) 95% credible interval of Θ.

5.4 Let Y_n be the nth order statistic of a random sample of size n from a distribution with pdf $f(x|\theta) = \frac{1}{\theta}, 0 < x \leq \theta$, zero elsewhere. Take the loss function to be $\mathcal{L}[\theta, \delta(y)] = [\theta - \delta(y_n)]^2$. Let θ be an observed value of the random variable Θ, which has pdf $h(\theta) = \frac{\beta\alpha^\beta}{\theta^{\beta+1}}, \alpha < \theta < \infty$, zero elsewhere, with $\alpha > 0, \beta > 0$. Find the Bayes' solution $\delta(y_n)$ for a point estimate of θ and 90% credible interval of Θ.

5.5 In Exercise 5.5, let $n = 4$ from the uniform pdf $f(x, \theta) = \frac{1}{\theta}. 0 < x < \theta$, and the prior pdf be $g(\theta) = \frac{2}{\theta^3}, 1 < \theta < \infty$, zero elsewhere. Find:
(a) The Bayesian estimator $\delta(Y_4)$ of θ, based upon the sufficient statistic Y_4, using the loss function $[\theta - \delta(y_4)]^2$.
(b) The Bayesian estimator $\delta(Y_4)$ of θ, based upon the sufficient statistic Y_4, using the loss function $|\delta(y_4) - \theta|$.
(c) 98% credible interval of Θ.

5.6 Consider the model

$$X_i|\delta \sim \text{iid } Exponential\left(\frac{1}{\delta}\right)$$
$$\delta \sim Gamma\left(\alpha, \frac{1}{\beta}\right)$$

Find the following:

(a) Posterior distribution of δ.
(b) Bayes' point estimate of δ in different two ways.

(c) 95% credible interval of δ.

(d) If $X_1 = 2.5, X_2 = 3.61, X_3 = 4.8, X_4 = 2.74, X_5 = 3.95$ and $\alpha = 2, \beta = 4$. Calculate (b) and (c).