King Saud University

College of Sciences

Department of Mathematics

151 Math Exercises

(3,2)

Methods of Proof

"Mathematical Induction"

(First principle)

Malek Zein AL-Abidin

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Mathematical Induction

In general, mathematical induction ∗ can be used to prove statements that assert that *P(n)* is true for all positive integers *n*, where *P(n)* is a propositional function. A proof by mathematical induction has two parts, a **basis step**, where we show that *P*(1) is true, and an **inductive step**, where we show that for all positive integers *k*, if *P(k)* is true, then *P*(*k* + 1) is true.

> *PRINCIPLE OF MATHEMATICAL INDUCTION* To prove that *P*(*n*) is true for all positive integers *n*, where *P*(*n*) is a propositional function, we complete two steps:
> *BASIS STEP:* We verify that *P*(1) is true.
>
> *INDUCTIVE STEP:* We show that the conditional statement $P(k) \to P(k+1)$ is true for all positive integers *k*.

Exercises

1. Use mathematical induction to Show that

if *n* is a positive integer, then $1 + 2 + \cdots + n = \frac{n(n+1)}{2}$

Solution: Let *P*(*n*) be the proposition , $P(n)$: $1 + 2 + \cdots + n = \frac{n(n+1)}{2}$, $\forall n \geq 1$

BASIS STEP: *P*(1) , L H S = 1 , R H S = $\frac{1(1+1)}{2} = 1$

∵ L H S = R H S ⇒ ∴ *P*(1) is true .

INDUCTIVE STEP: We assume that *P*(*k*) holds for an arbitrary positive integer *k*. That is, we assume that $P(k)$: $1 + 2 + \cdots + k = \frac{k(k+1)}{2}$: $k \geq 1$, is true .

Under this assumption, it must be shown that *P*(*k* + 1) is also true . When we add(*k* + 1) to both sides of the equation in *P*(*k*), we obtain

$$P(k+1):\ 1 + 2 + \cdots + k + (k+1) = \frac{k(k+1)}{2} + (k+1)$$

$$= \frac{k(k+1)+2(k+1)}{2}$$

$$= \frac{k^2+3k+2}{2}$$

$$= \frac{(k+1)(k+2)}{2}$$

This last equation shows that *P*(*k* + 1) is true under the assumption that *P*(*k*) is true. This completes the inductive step. *P*(*k*+1) is true . Then *P*(*n*) is true for $\forall n \geq 1$.

2. Use mathematical induction to Show that

$$\frac{1}{1\times 3}+\frac{1}{3\times 5}+\frac{1}{5\times 7}+\cdots+\frac{1}{(2n-1)(2n+1)}=\frac{n}{2n+1} \quad : \forall n\geq 1$$

Solution:

3. Use mathematical induction to Show that

$$1.2.3 + 2.3.4 + \cdots + n(n+1) = \frac{n(n+1)(n+2)}{3} : \ \forall n \geq 1$$

Solution:

4. Use mathematical induction to Show that

$$1.2.3 + 2.3.4 + \cdots + n(n+1)(n+2) = \frac{n(n+1)(n+2)(n+3)}{4} : \ \forall n \geq 1$$

Solution:

5. Use mathematical induction to Show that
if n is a positive integer, then $1 + 3 + 5 + \cdots + (2n - 1) = n^2$

Solution: Let $P(n)$ be the proposition ,

$$P(n):\ 1 + 3 + 5 + \cdots + (2n - 1) = n^2 \quad ,\forall n \geq 1$$

BASIS STEP: $P(1)$: L H S = 1 , R H S = $1^2 = 1$

$\because$ L H S = R H S $\Rightarrow \therefore P(1)$ is true .

INDUCTIVE STEP: We assume that $P(k)$ holds for an arbitrary positive integer k. That is, we assume that $P(k):\ 1 + 3 + 5 + \cdots + (2k - 1) = k^2 \ : k \geq 1$, is true .

Under this assumption, it must be shown that $P(k+1)$ is also true . When we add $(2k + 1)$ to both sides of the equation in $P(k)$, we obtain

$$P(k+1):\ 1 + 3 + 5 + \cdots + (2k - 1) + (2k+1) = k^2 + (2k + 1)$$

(where $P(k)$ is true) [the term # $n = k+1$]

$$= k^2 + 2k + 1$$
$$= (k + 1)^2$$

$$\therefore \quad P(k+1):\ 1 + 3 + 5 + \cdots + (2k - 1) + (2k+1) = (k + 1)^2$$

This last equation shows that $P(k + 1)$ is true under the assumption that $P(k)$ is true. This completes the inductive step. $P(k+1)$ is true . Then $P(n)$ is true for $\forall n \geq 1$.

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6. Use mathematical induction to Show that

$$\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\cdots+\frac{1}{2^n}=\frac{2^n-1}{2^n} \qquad : \forall n \geq 1$$

Solution:

7. Use mathematical induction to Show that

$$1 + 2 + 2^2 + \cdots + 2^n = 2^{n+1} - 1$$

for all nonnegative integers n.

Solution: Let $P(n)$ be the proposition ,

$$P(n):\ 1 + 2 + 2^2 + \cdots + 2^n = 2^{n+1} - 1 \quad , \forall n \geq 0$$

BASIS STEP: $P(0)$: L H S $= 2^0 = \boxed{1}$, R H S $= 2^{0+1} - 1 = 2 - 1 = \boxed{1}$

$\because$ L H S = R H S $\Rightarrow \therefore P(0)$ is true .

INDUCTIVE STEP: We assume that $P(k)$ holds for an arbitrary nonnegative integer k. That is, we assume that

$$P(k):\ 1 + 2 + 2^2 + \cdots + 2^k = 2^{k+1} - 1 : k \geq 0 \quad , \text{is true} .$$

Under this assumption, it must be shown that $P(k+1)$ is also true . When we add 2^{k+1} to both sides of the equation in $P(k)$, we obtain

$$P(k+1):\ 1 + 2 + 2^2 + \cdots + 2^k + (2^{k+1}) = 2^{k+1} - 1 + 2^{k+1}$$

(where $P(k)$ is true) [the term # $n = k+1$]

$$= 2\ 2^{k+1} - 1$$

$$= 2^{k+2} - 1$$

$$= 2^{(k+1)+1} - 1$$

$$\therefore \quad P(k+1):\ 1 + 2 + 2^2 + \cdots + 2^k + (2^{k+1}) = 2^{(k+1)+1} - 1$$

This last equation shows that $P(k+1)$ is true under the assumption that $P(k)$ is true. This completes the inductive step. $P(k+1)$ is true . Then $P(n)$ is true for $\forall n \geq 0$.

8. Use mathematical induction to Show that

$$2 + 2(-7) + 2(-7)^2 + 2(-7)^3 + \cdots + 2(-7)^n = \frac{1 - (-7)^{n+1}}{4} : n \geq 0$$

Solution:

9. Use mathematical induction to Show that

$$1 \times 2 + 2 \times 2^2 + 3 \times 2^3 + \cdots + n \times 2^n = 2 + (n-1)2^{n+1} \ : n \geq 1$$

Solution:

10. Use mathematical induction to Show that

$$\sum_{j=0}^{n} ar^j = a + ar + ar^2 + \cdots + ar^n = \frac{ar^{n+1} - a}{r - 1}$$

where n is a nonnegative integer. , when $r \neq 1$

Solution: Let $P(n)$ be the proposition ,

$P(n)$:

$$\sum_{j=0}^{n} ar^j = a + ar + ar^2 + \cdots + ar^n = \frac{ar^{n+1} - a}{r - 1}$$

Where $r \neq 1$, $\forall n \geq 0$

BASIS STEP: $P(0)$: L H S = $ar^0 = a.1 = a$, R H S = $\frac{ar^{0+1}-a}{r-1} = \frac{a(r-1)}{r-1} = a$

$\because$ L H S = R H S $\Rightarrow \therefore P(0)$ is true .

INDUCTIVE STEP: We assume that $P(k)$ holds for an arbitrary nonnegative integer k. That is, we assume that

$P(k)$: $a + ar + ar^2 + \cdots + ar^k = \frac{a\,r^{k+1}-a}{r-1}$: $k \geq 0$, is true .

Under this assumption, it must be shown that $P(k+1)$ is also true . When we add ar^{k+1} to both sides of the equation in $P(k)$, we obtain

$P(k+1)$: $a + ar + ar^2 + \cdots + ar^k + (ar^{k+1}) = \frac{a\,r^{k+1}-a}{r-1} + ar^{k+1}$

(where $P(k)$ is true) [the term # $n = k+1$]

$\Rightarrow$

$$\frac{ar^{k+1} - a}{r - 1} + ar^{k+1} = \frac{ar^{k+1} - a}{r - 1} + \frac{ar^{k+2} - ar^{k+1}}{r - 1}$$

$$= \frac{ar^{k+2} - a}{r - 1}.$$

Combining these last two equations gives

$$a + ar + ar^2 + \cdots + ar^k + ar^{k+1} = \frac{ar^{k+2} - a}{r - 1}.$$

This last equation shows that $P(k+1)$ is true under the assumption that $P(k)$ is true. This completes the inductive step . $P(k+1)$ is true . Then $P(n)$ is true for $\forall n \geq 0$.

11. Use mathematical induction to Show that

$$1 + a + a^2 + \cdots + a^n = \frac{a^{n+1} - 1}{a-1} \quad , a \neq 1$$: for all nonnegative integers n.

Solution:

12. Use mathematical induction to Show that

$$2 + 4 + 6 + \cdots + 2n = n(n + 1) \quad : n \geq 1$$

Solution:

13. Use mathematical induction to Show that

$$1 + 4 + 7 + \cdots + (3n - 2) = \frac{n(3n - 2)}{2} \qquad : n \geq 1$$

Solution:

14. Use mathematical induction to Show that

$$3 + 3^2 + 3^3 + \cdots + 3^n = \frac{3}{2}(3^n - 1) : n \geq 1$$

Solution:

15. Use mathematical induction to Show that

$$\frac{1}{2}+\frac{2}{2^2}+\frac{3}{2^3}+\cdots+\frac{n}{2^n}=2-\frac{n+2}{2^n} \quad : n \geq 1$$

Solution:

16. Use mathematical induction to Show that

$$2 + 2 \times 3 + 2 \times 3^2 + 2 \times 3^3 + \cdots + 2 \times 3^n = 3^n - 1 : n \geq 1$$

Solution:

17. Use mathematical induction to Show that

$$1^2 + 3^2 + 5^2 + \cdots + (2n-1)^2 = \frac{n(2n-1)(2n+1)}{3} : \forall n \geq 1$$

Solution:

18. Use mathematical induction to Show that

$$1^3 + 2^3 + 3^3 + \cdots + n^3 = [\frac{n(n+1)}{2}]^2 \quad : \qquad \forall n \geq 1$$

Solution:

19. Use mathematical induction to Show that

$$n < 2^n \quad : \forall n \geq 1$$

Solution: Let $P(n)$ be the proposition ,

$P(n)$: $n < 2^n \quad : \forall n \geq 1$

BASIS STEP: $P(1)$: $1 < 2^1 = 2$, $\therefore$ $P(1)$ is true

INDUCTIVE STEP: We assume that $P(k)$ holds for an arbitrary positive integer k. That is, we assume that $P(k)$: $k < 2^k \quad : k \geq 1$, is true .

Under this assumption, it must be shown that $P(k+1)$ is also true .

$P(k+1)$: $k+1 < 2^k + 1 < 2^k + 2^k = 2\ 2^k = 2^{k+1}$ where $1 < 2^k$

$\therefore$ $k+1 < 2^{k+1} \Rightarrow P(k+1)$ is true . Then $P(n)$ is true for $\forall n \geq 1$.

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20. Use mathematical induction to Show that

$2^n < n!$ for every integer n with $n \geq 4$.

Solution: Let $P(n)$ be the proposition ,

$P(n)$: $2^n < n! \quad : \forall n \geq 4$

BASIS STEP: $P(4)$: $2^4 = 16 < 4! = 24$, $\therefore$ $P(4)$ is true

INDUCTIVE STEP: We assume that $P(k)$ holds for an arbitrary positive integer $k : k \geq 4$.

That is, we assume that $P(k)$: $2^k < k! \quad : k \geq 4$, is true . (*)

Under this assumption, it must be shown that $P(k+1)$ is also true .

$P(k+1)$: $2^{k+1} = 2\ 2^k < 2\ k!$ (from inductive hypothesis *)

$< (k+1)\ k!$ (because $2 < k+1$)

$= (k+1)!$ (by definition of factorial function)

$\therefore$ $2^{k+1} < (k+1)! \Rightarrow P(k+1)$ is true . Then $P(n)$ is true for $\forall n \geq 4$.

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21. Use mathematical induction to Show that

$$3^n < n! \quad \text{for every integer } n \text{ with } n \geq 7.$$

Solution:

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22. Use mathematical induction to Show that

$$n! < n^n \qquad \text{for every integer } n \text{ with } n \geq 2.$$

Solution:

23. Use mathematical induction to Show that

$$2^n \geq n + 12 \quad \text{for every integer } n \text{ with } n \geq 4.$$

Solution:

24. Use mathematical induction to Show that

$$2^n > n^2 \quad \text{for every integer } n \text{ with } n \geq 5.$$

Solution:

25. Use mathematical induction to Show that

$$n^2 > 4 + n \quad \text{for every integer } n \text{ with } n \geq 3.$$

Solution: Let $P(n)$ be the proposition ,

$$p(n): \quad n^2 > 4 + n \ : \forall\, n \geq 3.$$

BASIS STEP: $p(3)$: $3^2 = 9 > 4 + 3 = 7 \quad \Rightarrow \therefore\ p(3)$ *is true*

INDUCTIVE STEP: We assume that $P(k)$ holds for an arbitrary positive integer k. That is,

we assume that $p(k): \ k^2 > 4 + k \quad : k \geq 3$, is true . (*)

under this assumption, it must be shown that $P(k+1)$ is also true .

$$p(k+1): \ (k+1)^2 = k^2 + 2k + 1$$

$$> 4 + k + 2k + 1 \qquad \text{(from inductive hypothesis *)}$$

$$> 4 + k + 1 \qquad \text{(because } 2k > 1 \ : k \geq 3 \text{)}$$

$$= 4 + (k+1)$$

$$\therefore\ (k+1)^2 > 4 + (k+1) \ \Rightarrow \therefore\ P(k+1) \text{ is true .}$$

Then $P(n): \ n^2 > 4 + n \quad : \forall n \geq 3$ is true .

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26. Use mathematical induction to Show that

$$2^n > n^2 + 19 \qquad \text{for every integer } n \text{ with } n \geq 5.$$

Solution:

27. Use mathematical induction to Show that

$$n^3 > 2n + 1 \qquad \text{for every integer } n \text{ with } n \geq 2.$$

Solution:

28. Use mathematical induction to Show that

$$\frac{1}{\sqrt{1}}+\frac{1}{\sqrt{2}}+\cdots+\frac{1}{\sqrt{n}} \geq \sqrt{n} \qquad \text{for every integer } n \text{ with } n \geq 1.$$

Solution: Let $P(n)$ be the proposition ,

$$p(n): \quad \frac{1}{\sqrt{1}}+\frac{1}{\sqrt{2}}+\cdots+\frac{1}{\sqrt{n}} \geq \sqrt{n} \quad : \forall\, n \geq 1.$$

BASIS STEP: $p(1)$: $\frac{1}{\sqrt{1}} = 1 \geq \sqrt{1} \Rightarrow \therefore \; p(1)$ *is true* .

INDUCTIVE STEP: We assume that $P(k)$ holds for an arbitrary positive integer k. That is,

we assume that $p(k)$: $\frac{1}{\sqrt{1}}+\frac{1}{\sqrt{2}}+\cdots+\frac{1}{\sqrt{k}} \geq \sqrt{k} \quad : k \geq 1$, is true . (*)

under this assumption, it must be shown that $P(k+1)$ is also true .

$$p(k+1): \left[\frac{1}{\sqrt{1}}+\frac{1}{\sqrt{2}}+\cdots+\frac{1}{\sqrt{k}}\right]+\frac{1}{\sqrt{k+1}} \geq \left[\sqrt{k}\,\right]+\frac{1}{\sqrt{k+1}}$$ (from inductive hypothesis *)

$$= \frac{\sqrt{k}\,\sqrt{k+1}+1}{\sqrt{k+1}}$$

$$= \frac{\sqrt{k(k+1)}+1}{\sqrt{k+1}}$$

$$\geq \frac{\sqrt{k(k)}+1}{\sqrt{k+1}} = \frac{\sqrt{k^2}+1}{\sqrt{k+1}}$$ (because $k+1 > k$)

$$= \frac{k+1}{\sqrt{k+1}} = \sqrt{k+1}$$

$$\therefore \quad \frac{1}{\sqrt{1}}+\frac{1}{\sqrt{2}}+\cdots+\frac{1}{\sqrt{k}}+\frac{1}{\sqrt{k+1}} \geq \sqrt{k+1}$$

$\therefore p(k+1)$ is true .

Then $P(n)$: $\frac{1}{\sqrt{1}}+\frac{1}{\sqrt{2}}+\cdots+\frac{1}{\sqrt{n}} \geq \sqrt{n} \quad : \forall n \geq 1$. is true .

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29. Use mathematical induction to Show that

$$\frac{1}{\sqrt{1}}+\frac{1}{\sqrt{2}}+\cdots+\frac{1}{\sqrt{n}}> 2(\sqrt{n+1}-1)$$ for every integer n with $n \geq 1$.

Solution:

30. Use mathematical induction to Show that

$$\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\cdots+\frac{1}{2^n} < 1 \qquad \text{for every integer } n \text{ with } n \geq 1.$$

Solution:

31. Use mathematical induction to Show that

$$n^2 - 7n + 12 \geq 0 \qquad \text{for every integer } n \text{ with } n \geq 3.$$

Solution:

32. Use mathematical induction to Show that

$$1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{n^2} < 2 - \frac{1}{n} \qquad \text{for every integer } n \text{ with } n \geq 2.$$

Solution:

33. Use mathematical induction to Show that

$$3^n \geq 2^{n+2} \qquad \text{for every integer } n \text{ with } n \geq 4.$$

Solution:

34. Use mathematical induction to Show that

$$n^2 - 3n + 5 \quad \text{is odd} \quad \text{for all nonnegative integers } n.$$

Solution:

35. Use mathematical induction to Show that

$$3 \mid 4^n - 1 \qquad \text{for all nonnegative integers } n.$$

Solution: Let $P(n)$ be the proposition ,

$$p(n): \quad 3 \mid 4^n - 1 \quad : \forall\, n \geq 0 \Rightarrow \exists c \in \mathbb{Z} : 4^n - 1 = 3c$$

BASIS STEP: $p(0)$: $\quad 3 \mid 4^0 - 1 \Rightarrow 3 \mid 1 - 1 = 0 \Rightarrow 3 \mid 0 \Rightarrow \therefore \quad p(0)$ *is true* .

INDUCTIVE STEP: We assume that $P(k)$ holds for an arbitrary nonnegative integer k.

That is, we assume that $\quad p(k): \quad 3 \mid 4^k - 1 \quad : k \geq 0$, is true .

$$\Rightarrow \quad 4^k - 1 = 3c \quad \Rightarrow \quad 4^k = 3c + 1 \qquad (*)$$

under this assumption, it must be shown that $P(k+1)$ is also true .

$$p(k+1): \quad 4^{k+1} - 1 = 4\, 4^k - 1$$

$$= 4\,(3c + 1) - 1 \qquad \text{(from inductive hypothesis } * \text{)}$$

$$= 12c + 4 - 1 = 12c + 3$$

$$= 3(4c + 1) = 3h \qquad : h = (4c + 1) \in \mathbb{Z}$$

$$\Rightarrow \quad 3 \mid 4^{k+1} - 1$$

$\therefore \quad p(k+1)$ is true

$\therefore \quad p(n)$ *is true for* $\forall\, n \geq 0$.

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36. Use mathematical induction to Show that

$$3|(n^3 + 2n) \qquad \text{for all positive integer } n$$

Solution:

37. Use mathematical induction to Show that

$$5|7^n - 2^n \qquad \forall\ n \geq 1$$

Solution:

38. Use mathematical induction to Show that

$$3|5^{n} - 2^{n+2} \qquad \text{for all nonnegative integers } n.$$

Solution:

39. Use mathematical induction to Show that

$$7 | 9^{2n} - 5^{2n} \qquad \forall\ n \geq 1$$

Solution:

40. Use mathematical induction to prove that $n^3 - n$ is divisible by 3 whenever n is a positive integer .

$$3 | n^3 - n \qquad \forall \ n \geq 1$$

Solution:

41. Use mathematical induction to prove that $n(n^2 + 5)$ is divisible by 6 whenever n is a positive integer .

$$6 | n(n^2 + 5) \qquad \forall \; n \geq 1$$

Solution:

42. Use mathematical induction to prove that $n^3 - n + 3$ is divisible by 3 whenever n is a nonnegative integer .

$$3 \mid n^3 - n + 3 \qquad \forall \; n \geq 0$$

Solution:

43. Use mathematical induction to Show that

$$5 \mid 2^{2n-1} + 3^{2n-1} \qquad \forall\ n \geq 1$$

Solution:

44. Use mathematical induction to Show that

$$2 \mid n^2 + n \qquad \forall\ n \geq 0$$

Solution:

45. Use mathematical induction to prove that $n^5 - n$ is divisible by 5 whenever n is a nonnegative integer .

$$5 | n^5 - n \qquad \forall \; n \geq 0$$

Solution:

46. Prove that 21 divides $4^{n+1} + 5^{2n-1}$ whenever n is a positive integer .

Solution:

47. Prove that if n is a positive integer, then 133 divides $11^{n+1} + 12^{2n-1}$

Solution:

48. Use mathematical induction to prove that $7^{n+2} + 8^{2n+1}$ is divisible by 57 for every nonnegative integer n.

Solution: