Question1. (a) Are the functions $f_1(x) = \cos 2x,\ f_2(x) = 1,\ f_3(x) = \cos^2 x$ linearly independent on $\mathbb{R}$?

(b) Determine the constants $\beta_i,\ i = 1, 2, 3, 4, 5$ in the function

$$F(x) = \beta_1 + \beta_2 \sin x + \beta_3 \cos x + \beta_4 \sin 2x + \beta_5 \cos 2x,$$

to obtain the best approximation for $F(x) = |x|$ in $\mathfrak{L}^2(-\pi, \pi)$.

(c) Which of the following functions belongs to $\mathfrak{L}^2(2, \infty)$

i. $f(x) = \frac{1}{\sqrt{x^2+1}}$

ii. $g(x) = \sin x$

iii. $h(x) = \frac{1}{\sqrt{x \ln x}}$

Question2. (a) Obtain the eigenvalues and eigenfunctions of the problem

$$\begin{aligned} u'' + \lambda u &= 0,\ x \in (-2, 2) \\ u(-2) &= u(2) \\ u'(-2) &= u'(2) \end{aligned}$$

Is the above problem a Sturm-Liouville problem? Explain.

(b) Write the orthogonality relation between the eigenfunctions of the above problem.

(c) Let $P_n(x)$ be the Legendre polynomials orthogonal on $[-1, 1]$. Find the expansion of $f(x) = |2x - 1| - |x|,\ |x| < 1$ in terms of $P_n(x)$.

Question3. (a) Find the Fourier Series for $f(x) = |x| - x,\ -1 < x < 1$ such that $f(x + 2) = f(x)$.Then deduce that

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} = \frac{\pi^2}{8}$$

(b) Find the Fourier Integral for the function

$$f(x) = \begin{cases} 0, & |x| > \pi \\ |\cos x| & |x| \leq \pi \end{cases}$$

(c) Solve the integral equation

$$\int_0^{\infty} f(\xi) \sin(x\xi) d\xi = \begin{cases} x, & 0 < x < 1 \\ 0, & x > 1 \end{cases}$$

Question4. (a) Find the Fourier transform of

$$f(x) = \begin{cases} 1 - x^2, & |x| < 1 \\ 0, & |x| > 1 \end{cases}$$

and deduce the value of

$$\int_0^\infty \frac{(x \cos x - \sin x) \cos x}{x^3} dx$$

(b) Show that

$$\frac{d}{dx}(x^n J_n(x)) = x^n J_{n-1}(x),$$

where $J_n(x)$ is the Bessel functions of the first kind.

Answer sheet

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Q1 a) $f_3(x) = \cos^2 x = \frac{1+\cos 2}{2} = \frac{f_2}{2} + \frac{f_1}{2} \quad \forall x \in \mathbb{R}$

That is $\frac{1}{2} f_1(x) + \frac{1}{2} f_2(x) - f_3(x) = 0 \quad \forall x \in \mathbb{R}$

$c_1 = \frac{1}{2}, \quad c_2 = \frac{1}{2}, \quad c_3 = -1$

So the functions are not linearly independent on $\mathbb{R}$.
But they are linearly dependent on $\mathbb{R}$.

b) $F(x) = |x| \in L^2(-\pi, \pi)$ since $\int_{-\pi}^{\pi} x^2 dx = 2\int_0^{\pi} x^2 dx = \frac{2\pi^3}{3} < \infty$

$F(x) = \sum_{n=0}^{4} \frac{\langle f, \varphi_n \rangle}{\|\varphi_n\|^2} \cdot \varphi_n(x)$ with (φ_n) is the is the orthogonal set of function $\{1, \sin x, \cos x, \sin 2x, \cos 2x\}$

$\langle f, \varphi_0 \rangle = \langle |x|, 1 \rangle = \int_{-\pi}^{\pi} |x| dx = 2\int_0^{\pi} x dx = \pi^2$

$\langle f, \varphi_1 \rangle = \langle |x|, \sin x \rangle = \int_{-\pi}^{\pi} |x| \sin(x) dx = 0$ (integral of odd function over a symmetric interval)

$\langle f, \varphi_2 \rangle = \langle |x|, \cos x \rangle = \int_{-\pi}^{\pi} |x| \cos x \, dx = 2\int_0^{\pi} x \cos x \, dx = -2$

$\langle f, \varphi_3 \rangle = \langle |x|, \sin 2x \rangle = 0$

$\langle f, \varphi_4 \rangle = \langle |x|, \cos 2x \rangle = 2\int_0^{\pi} x \cos 2x \, dx = 0$

~~Then F(x) = |x| =~~ $\|\varphi_0\|^2 = \int_{-\pi}^{\pi} dx = 2\pi$

$\|\varphi_i\|^2 = \int_{-\pi}^{\pi} \varphi_i^2 dx = \pi, \quad i = 1, 2, 3, 4$

Then $F(x) = |x| = \frac{\pi^2}{2\pi} \cdot 1 + 0 \cdot \sin x - \frac{2}{\pi} \cos x + 0 \cdot \sin 2x + 0 \cos 2x$

So $\beta_1 = \frac{\pi}{2}, \; \beta_2 = 0, \; \beta_3 = -\frac{2}{\pi}, \; \beta_4 = 0, \; \beta_5 = 0$

c) i) $\int_2^{\infty} \frac{dx}{x^2+1} = \tan^{-1} x \Big|_2^{\infty} = \frac{\pi}{2} - \tan^{-1} 2 < \infty \Rightarrow f \in L^2(2, \infty)$

ii) $\int_2^{\infty} \sin^2 x \, dx = \int_2^{\infty} \frac{1 - \cos 2x}{2} dx = \frac{x}{2}\Big|_2^{\infty} - \frac{\sin 2x}{4}\Big|_2^{\infty} = \infty \Rightarrow g \notin L^2(2, \infty)$

iii) $\int_2^{\infty} \frac{dx}{x \ln x} = \int_2^{\infty} \frac{d(\ln x)}{\ln x} = \frac{(\ln x)^2}{2}\Big|_2^{\infty} = \infty \Rightarrow h(x) \notin L^2(2, \infty)$

Q2 a): The boundary conditions are not of the form

$$\begin{cases} \alpha_1 u(-2) + \alpha_2 u'(2) = 0 \\ \beta_1 u(-2) + \beta_2 u'(2) = 0 \end{cases}$$

(2)

Thus the problem (*) is not a Sturm-Liouville.

1) $\lambda > 0$: The characteristic eq: $m^2 + \lambda = 0 \Rightarrow m = \pm i\sqrt{\lambda}$

$u(x) = C_1 \cos(\sqrt{\lambda}\, x) + C_2 \sin(\sqrt{\lambda}\, x)$

$u(-2) = u(2) \Rightarrow \cancel{C_1 \cos(2\sqrt{\lambda})} - C_2 \sin(\sqrt{\lambda}\, 2) = \cancel{C_1 \cos(\sqrt{\lambda}\, 2)} + C_2 \sin(2\sqrt{\lambda})$

$\Rightarrow 2 C_2 \sin(\sqrt{\lambda}\, 2) = 0$

If $C_2 = 0$, then $u(x) = C_1 \cos(\sqrt{\lambda}\, x)$

$u'(x) = -C_1 \sqrt{\lambda} \sin(\sqrt{\lambda}\, x)$

$u'(-2) = u'(2) \Rightarrow C_1 \sqrt{\lambda} \sin(2\sqrt{\lambda}) = -C_1 \sqrt{\lambda} \sin(2\sqrt{\lambda})$

$\Rightarrow 2 C_1 \sqrt{\lambda} \sin(2\sqrt{\lambda}) = 0$

If $C_1 = 0$, then $u(x) = 0$ (trivial sol)

So $C_1 \neq 0$, then $\sin(2\sqrt{\lambda}) = 0 = \sin(n\pi)$

Thus $\lambda_n = \left(\frac{n\pi}{2}\right)^2$, $n \geq 1$ (eigenvalues).

The eigenfunctions are $u_n(x) = \left(\cos\left(\frac{n\pi}{2} x\right)\right)_{n \geq 1}$

2) $\lambda = 0 \Rightarrow u'' = 0 \Rightarrow u(x) = C_1 x + C_2$

$u(-2) = -2C_1 + C_2 = u(2) = 2C_1 + C_2 \Rightarrow 4C_1 = 0 \Rightarrow C_1 = 0$

Hence $u(x) = C_2$

$u'(x) = C_1 = 0 \Rightarrow u'(-2) = u'(2)$

We can take $u(x) = 1$ (which satisfies the BCs)

b) $(u_n, u_m) = \int_{-2}^{2} \cos\left(\frac{n\pi x}{2}\right) \cos\left(\frac{m\pi x}{2}\right) dx$

$= \frac{1}{2} \int_{-2}^{2} \left[\cos(n+m)\frac{\pi x}{2} + \cos(n-m)\frac{\pi x}{2}\right] dx$

$= \frac{1}{2} \left[\frac{\sin(n+m)\frac{\pi x}{2}}{(n+m)\frac{\pi}{2}} \Big|_{-2}^{2} + \frac{\sin(n-m)\frac{\pi x}{2}}{(n-m)\frac{\pi}{2}} \Big|_{-2}^{2} \right] = 0 \quad \forall n \neq m$

c) $f(x) = |2x-1| - |x|$ for $|x|<1$

$$f(x) = \begin{cases} x-1, & \frac{1}{2} < x \le 1 \\ 1-3x, & 0 < x < \frac{1}{2} \\ 1-x, & -1 \le x \le 0 \end{cases}$$

$$f(x) = \sum_{n=0}^{\infty} \frac{\langle f, P_n \rangle}{\|P_n\|^2} P_n(x) = \frac{\langle f, P_0 \rangle}{\|P_0\|^2} P_0(x) + \frac{\langle f, P_1 \rangle}{\|P_1\|^2} P_1(x) + \cdots\cdots$$

$$\frac{\langle f, P_0 \rangle}{\|P_0\|^2} = \frac{1}{2}\int_{-1}^{0} (1-x)\,dx + \frac{1}{2}\int_{0}^{\frac{1}{2}} (1-3x)\,dx + \frac{1}{2}\int_{\frac{1}{2}}^{1} (x-1)\,dx = \frac{3}{4} \qquad (\|P_0\|^2 = 2)$$

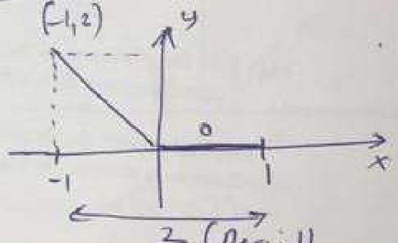
$$\frac{\langle f, P_1 \rangle}{\|P_1\|^2} = \frac{3}{2}\int_{-1}^{0} (1-x)x\,dx + \frac{3}{2}\int_{0}^{\frac{1}{2}} (1-3x)x\,dx + \frac{3}{2}\int_{\frac{1}{2}}^{1} (x-1)x\,dx \qquad \left(\|P_1\|^2 = \frac{2}{3}\right)$$

$$= -\frac{11}{8}$$

Thus $f(x) = \frac{3}{4} P_0(x) - \frac{11}{8} P_1(x) + \cdots\cdots$

Q_3 a) $f(x) = |x| - x, \; -1 < x < 1$

$$= \begin{cases} -2x & -1 < x < 0 \\ 0, & 0 \le x < 1 \end{cases}$$



It is clear that f, and f' are piece wise continuous, and since $f(x+2) = f(x)$, then

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\pi x) + b_n \sin(n\pi x)$$

$$a_0 = \frac{1}{2}\int_{-1}^{1} f(x)\,dx = \frac{-1}{2}\int_{-1}^{0} 2x\,dx = -\frac{x^2}{2}\Big|_{-1}^{0} = \frac{1}{2}$$

$$a_n = \frac{1}{2}\int_{-1}^{1} f(x)\cos(n\pi x)\,dx = \frac{-1}{2}\int_{-1}^{0} 2x\cos(nx\pi)\,dx =$$

$$= -\left[\frac{x\sin(n\pi x)}{n\pi}\Big|_{-1}^{0} - \int_{-1}^{0} \frac{\sin(n\pi x)}{n\pi}\,dx\right]$$

$$= \frac{1}{n\pi}\left[\frac{-\cos(n\pi x)}{n\pi}\right]_{-1}^{0} = \frac{-1}{(n\pi)^2}\left[1 - (-1)^n\right] = \frac{(-1)^n - 1}{(n\pi)^2}$$

$$b_n = \frac{-1}{2}\int_{-1}^{0} 2x \sin(n\pi x)\,dx = -\left[\frac{-x\cos(n\pi x)}{n\pi}\Big|_{-1}^{0} + \int_{-1}^{0}\frac{\cos(n\pi x)}{n\pi}\,dx\right]$$ (4)

$$= \frac{(-1)^{n+1}}{n\pi} + \frac{\sin(n\pi x)}{n^2\pi^2}\Big|_{-1}^{0} = \frac{(-1)^{n+1}}{n\pi}$$

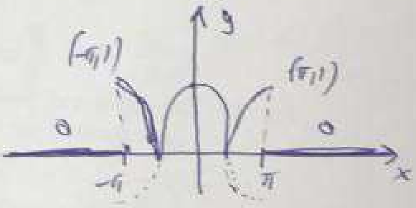
Thus $$f(x) = \frac{1}{4} + \sum_{n=1}^{\infty}\left[\frac{(-1)^n-1}{n^2\pi^2}\cos(n\pi x) + \frac{(-1)^{n+1}}{n\pi}\sin(n\pi x)\right]$$

$$= \frac{1}{4} - \frac{2}{\pi^2}\sum_{n=0}^{\infty}\frac{\cos(2n+1)\pi x}{(2n+1)^2} + \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin(n\pi x)$$

At $x=0$

$$f(0) = 0 = \frac{1}{4} - \frac{2}{\pi^2}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2} \Rightarrow \sum_{n=0}^{\infty}\frac{1}{(2n+1)^2} = \frac{\pi^2}{8}$$

b) $$f(x) = \begin{cases} 0, & |x|>\pi \\ |\cos x|, & |x|\le\pi \end{cases}$$



$f(-x)=f(x) \Rightarrow f$ is even on R.

f and f' are piecewise continuous on R. Thus

$$f(x) = \frac{1}{\pi}\int_0^{\infty} A(\xi)\cos(x\xi)\,d\xi \qquad B(\xi)=0$$

$$A(\xi) = 2\int_0^{\infty} f(x)\cos(x\xi)\,dx = 2\int_0^{\frac{\pi}{2}}\cos x\cos(x\xi)\,dx - 2\int_{\frac{\pi}{2}}^{\pi}\cos x\cos(x\xi)\,dx$$

$$= \int_0^{\frac{\pi}{2}}\left[\cos(1+\xi)x + \cos(1-\xi)x\right]dx - 2\int_{\frac{\pi}{2}}^{\pi}\left[\cos(1+\xi)x + \cos(1-\xi)x\right]dx$$

$$= \frac{2\cos(\frac{\pi\xi}{2}) - \xi\sin(\pi\xi)}{1-\xi^2}$$

Hence $$f(x) = \frac{1}{\pi}\int_0^{\infty}\left(\frac{2\cos(\frac{\pi\xi}{2}) - \xi\sin(\pi\xi)}{1-\xi^2}\right)\cos(x\xi)\,d\xi$$

Let $x=0$, we obtain $\displaystyle\int_0^{\infty}\frac{2\cos(\frac{\pi t}{2}) - t\sin(\pi t)}{1-t^2}\,dt = \pi$. $(\xi = t)$

c) If f is an odd function on $(-\infty, \infty)$, then

$$A(\xi) = \int_{-\infty}^{\infty} f(x)\cos(x\xi)\,dx = 0$$

$$B(\xi) = 2\int_0^{\infty} f(x)\sin(x\xi)\,dx$$

Then $$\int_0^{\infty} f(x)\sin(x\xi)\,dx = \begin{cases} 2\xi, & 0<\xi<1 \\ 0, & \xi>1 \end{cases}$$

and

$$f(x) = \frac{1}{\pi}\int_0^{\infty} B(\xi)\sin(x\xi)\,d\xi = \frac{2}{\pi}\int_0^{1} \xi\sin(x\xi)\,d\xi$$

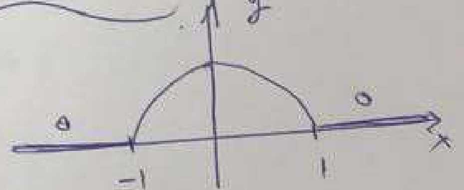
$$= \frac{2}{\pi}\left[-\frac{\xi\cos(x\xi)}{x}\Big|_0^1 + \int_0^1 \frac{\cos(x\xi)}{x}\,d\xi\right]$$

$$= \frac{-2}{\pi x}\left[\cos x\right] + \frac{2}{\pi x^2}\sin(x\xi)\Big|_0^1$$

$$= \frac{-2}{\pi}\frac{\cos x}{x} + \frac{2\sin x}{\pi x^2}$$

$$= \frac{2}{\pi}\left[\frac{\sin x}{x^2} - \frac{\cos x}{x}\right].$$

Q4 a) $$f(x) = \begin{cases} 1-x^2, & |x|<1 \\ 0, & |x|>1 \end{cases}$$



$f(-x) = f(x) \Rightarrow f$ is even on $\mathbb{R}$.

Then $$\hat{f}(\xi) = 2\int_0^{\infty} f(x)\cos(x\xi)\,dx$$

$$= 2\int_0^1 (1-x^2)\cos(x\xi)\,dx$$

$$= 2(1-x^2)\frac{\sin(x\xi)}{\xi}\Big|_0^1 + 4\int_0^1 \frac{x\sin(x\xi)}{\xi}\,dx$$

$$= \frac{4}{\xi}\left[\frac{-x\cos(x\xi)}{\xi}\Big|_0^1 + \int_0^1 \frac{\cos(x\xi)}{\xi}\,dx\right]$$

$$= \frac{-4\cos\xi}{\xi^2} + \frac{4}{\xi^3}\sin(x\xi)\Big|_0^1$$

$$= \frac{4}{\xi^3}\sin\xi - \frac{4}{\xi^2}\cos\xi = \frac{4\sin\xi - 4\xi\cos\xi}{\xi^3}.$$

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\xi)\cos(x\xi)\, d\xi$$ ⑥

$$= \frac{4}{\pi} \int_{0}^{\infty} \left(\frac{\sin\xi - \xi\cos\xi}{\xi^3} \right) \cos(x\xi)\, d\xi$$

Let $x = \frac{1}{2}$

$$f\left(\tfrac{1}{2}\right) = \frac{3}{4} = \frac{4}{\pi} \int_{0}^{\infty} \left(\frac{\sin(\xi) - \xi\cos\xi}{\xi^3} \right) \cos\left(\frac{\xi}{2}\right) d\xi$$

$$\Rightarrow \int_{0}^{\infty} \left(\frac{\sin(\xi) - \xi\cos(\xi)}{\xi^3} \right) \cos\left(\frac{\xi}{2}\right) d\xi = \frac{3\pi}{16}.$$

b) $$J_n(x) = \left(\frac{x}{2}\right)^n \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,\Gamma(m+n+1)} \left(\frac{x}{2}\right)^{2m}$$

then $$x^n J_n(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,(m+n+1)} \cdot \frac{x^{2m+2n}}{2^{2m+n}}$$

$$\frac{d}{dx}\left(x^n J_n(x)\right) = \sum_{m=0}^{\infty} \frac{(-1)^m (2m+2n)}{m!\,\Gamma(m+n+1)} \frac{x^{2m+2n-1}}{2^{2m+n}}$$

$$= x^n \sum_{m=0}^{\infty} \frac{(-1)^m (n+m)}{m!\,\Gamma(m+n+1)} \left(\frac{x}{2}\right)^{2m+n-1}$$

$$= x^n J_{n-1}(x).$$

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