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(8) اثبت ان الدالة $\psi(x) = e^{-x^2/2} H_n(x)$ تحقق المعادلة

$$\psi''(x) + [(2n+1) - x^2]\,\psi(x) = 0$$

معادلة شرودنجر

الاثبات:

$$\psi(x) = e^{-x^2/2} H_n(x)$$

$$\psi'(x) = -x e^{-x^2/2} H_n(x) + e^{-x^2/2} H_n'(x)$$

$$\psi''(x) = -e^{-x^2/2} H_n(x) + x^2 e^{-x^2/2} H_n(x) - x e^{-x^2/2} H_n'(x) - x e^{-x^2/2} H_n' + e^{-x^2/2} H_n''$$

$$\psi''(x) = -\psi(x) + x^2\psi(x) - 2x e^{-x^2/2} H_n'(x) + e^{-x^2/2}\Big[\underbrace{2x H_n'(x) - 2n H_n(x)}_{H_n''(x)}\Big]$$

$$= -\psi(x) + x^2\psi(x) - 2x e^{-x^2} e^{-x^2/2} H_n'(x) + 2x e^{-x^2} H_n'(x) - 2n e^{-x^2/2} H_n(x)$$

$$\psi'' = -\psi(x) + x^2\psi(x) - 2n\psi(x)$$

$$\psi'' = [-(2n+1) + x^2]\,\psi(x)$$

(5) عبر عن الدالة $f(x) = x^4$ بدلالة حدود هيرمت؟

$H_0(x) = 1$, $H_1(x) = 2x$, $H_2(x) = 4x^2 - 2$

$H_3(x) = 8x^3 - 12x$, $H_4(x) = 16x^4 - 48x^2 + 12$

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يتبع.

$$16x^4 = H_4(x) + 48x^2 + 12$$

$$\therefore x^4 = \frac{1}{16}H_4(x) + 3x^2 - \frac{3}{4}$$

$$H_2(x) = 4x^2 - 2 \Rightarrow 4x^2 = H_2(x) + 2 \Rightarrow x^2 = \frac{1}{4}H_2(x) + \frac{1}{2}$$

$$\Rightarrow x^4 = \frac{1}{16}H_4(x) + 3\left[\frac{1}{4}H_2(x) + \frac{1}{2}\right] - \frac{3}{4}H_0(x)$$

$$x^4 = \frac{1}{16}H_4(x) + \frac{3}{4}H_2(x) + \frac{3}{2}H_0(x) - \frac{3}{4}H_0(x)$$

$$x^2 = \frac{1}{16}H_4(x) + \frac{3}{4}H_2(x) + \frac{3}{4}H_0(x)$$

(4) أوجد منشور الدالة

$$f(x) = \begin{cases} 0 & ; \ x < 0 \\ 1 & ; \ x > 0 \end{cases}$$

بدلالة كثيرات حدود هيرميت أوجد مفكوك

$$f(x) = \sum_{n=0}^{\infty} c_n H_n(x)$$

$$\|H_n(x)\|^2 = 2^n n! \sqrt{\pi}$$

$$C_n = \frac{\langle f(x), H_n(x) \rangle}{\|H_n(x)\|^2}$$

$$\|H_0(x)\|^2 = 2^0\, 0!\, \sqrt{\pi} = \sqrt{\pi}$$

$$\|H_1(x)\|^2 = 2^1\, 1!\, \sqrt{\pi} = 2\sqrt{\pi}$$

$H_0(x) = 1, \ H_1(x) = 2x$

دالة الوزن $e^{-x^2}$

$$\langle f(x), H_0(x) \rangle = \int_{-\infty}^{\infty} f(x) H_0(x) e^{-x^2} dx = \int_0^{\infty} e^{-x^2} dx = \sqrt{\pi}/2$$

سيتم اثباته

$$C_0 = \frac{\sqrt{\pi}/2}{\sqrt{\pi}} = 1/2$$

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$$\langle f(x), H_1(x)\rangle = \int_0^\infty 2x\,e^{-x^2}dx \;\; \left(\lim_{t\to\infty}\int_0^t 2x\,e^{-x^2}dx\right) = \lim_{t\to\infty} -e^{-x^2}\Big|_0^t$$

$$= -\lim_{t\to\infty}\left[e^{-t^2} - e^{0}\right] = 0-(-1) = 1$$

$$\Rightarrow C_1 = \frac{1}{2\sqrt{\pi}}$$

$$\|H_2(x)\|^2 = 2^2\,2!\,\sqrt{\pi} = 8\sqrt{\pi}$$

$$H_2(x) = 4x^2-2$$

$$\langle f(x), H_2(x)\rangle = \int_0^\infty (4x^2-2)\,e^{-x^2}dx$$

$$= 4\int_0^\infty x^2 e^{-x^2}dx - 2\int_0^\infty e^{-x^2}dx =$$

$$= \underbrace{\lim_{t\to\infty} 4\int_0^t x^2 e^{-x^2}dx}_{I} - 2\,\frac{\sqrt{\pi}}{2} =$$

$$u = e^{-x^2} \Rightarrow du = -2x\,e^{-x^2}dx$$

$$dv = x^2dx \Rightarrow v = \frac{x^3}{3}$$

$$\Rightarrow I = \lim_{t\to\infty} 4\left[\frac{x^3}{3}e^{-x^2}\right]_0^t - \lim_{t\to\infty} 4\int_0^t \frac{x^3}{3}(-2x\,e^{-x^2})\,dx$$

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$$I = \lim_{t\to\infty} 4\int_0^t x^2 e^{-x^2}dx$$

$$= \lim_{t\to\infty} 2\int_0^t x\,2x\,e^{-x^2}dx$$

$$u = x \Rightarrow du = dx$$

$$dv = 2x\,e^{-x^2}dx \Rightarrow v = -e^{-x^2}$$

$$= \lim_{t\to\infty}\left[2\left[-x\,e^{-x^2}\right]_0^t - 2\int_0^t -e^{-x^2}dx\right]$$

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بحياته

$$= \sqrt{\pi/2}$$

$$= \frac{1}{2\sqrt{\pi}} \cdot [\sqrt{\pi} - \sqrt{\pi}] = 0$$

(11) عبر عن الدالة $f(x) = x^3 - x$ بدلالة كثيرات حدود لاجير.

$L_0(x) = 1, \quad L_1(x) = 1 - x$

$L_2(x) = 1 - 2x + \frac{1}{2}x^2, \quad L_3(x) = 1 - 3x + \frac{3}{2}x^2 - \frac{1}{6}x^3$

$$x^3 = 6 - 18x + 9x^2 - 6L_3(x)$$

$$x^3 = 6 - 18x + 9[2L_2(x) - 2 + 4x] - 6L_3(x)$$

$$x^3 = 6 - 18x + 18L_2(x) - 18 + 36x - 6L_3(x)$$

نطرح $x$ للطرفين

$$x^3 - x = \cancel{6} - 19x + 18L_2 - 12 + 36x - 6L_3(x)$$

$$x^3 - x = 17x + 18L_2 - 12 - 6L_3(x)$$

$$x^3 - x = 17(1 - L_1) + 18L_2 - 12 - 6L_3(x)$$

$$x^3 - x = 5 - 17L_1 + 18L_2 - 6L_3(x)$$

$\boxed{9}$ تحقق من تعامد الدوال $L_0, L_1, L_2, L_3$ على الفترة $[0, \infty)$ بالنسبة لدالة الثقل $e^{-x}$

$$\langle L_i, L_j \rangle = \int_0^\infty L_i L_j e^{-x} \, dx$$

$$L_n(x) = \frac{1}{n!} e^{x} \frac{d^n}{dx^n}(x^n e^{-x})$$

$$L_n(x) = \sum_{k=0}^{n} \frac{(-1)^k}{k!} \binom{n}{k} x^k$$

$$H_n(x) = \sum_{k=0}^{n/2} \frac{(-1)^k (2x)^{n-2k}}{k!\,(n-2k)!}$$

$$\int_0^\infty (1-x)e^{-x}dx = \lim_{t\to\infty}\left[\int_0^t e^{-x}dx - \int_0^t x e^{-x}dx\right]$$

$u=x$
$v=\frac{e^{-x}}{-1}$

$$= \lim_{t\to\infty}\left[\left[\frac{e^{-x}}{-1}\right]_0^t - \left[-xe^{-x}\right]_0^t + \int_0^t e^{-x}dx\right]$$

من الأفضل أن نجعل $u=(1-x)$ مسبقاً

$$= \lim_{t\to\infty}[+1-1] = 0$$

$$\langle L_0, L_2\rangle = \int_0^\infty (1-2x+\tfrac{1}{2}x^2)e^{-x}dx = 0$$

بالتجزيء ، $u=(1-2x+\frac{1}{2}x^2)$ وهكذا ... الأسئلة $(x)$

استفدنا من التمرين على حساب $\langle L_1, L_2\rangle$

$$x^n e^{-x}\Big|_0^\infty = 0 \qquad \int_0^\infty x^n e^{-x}dx < \infty \qquad \forall n\in\mathbb{N}$$

بالاستقراء الرياضي .

$$\langle L_i, L_j\rangle = \begin{cases} 0 : i\neq j \\ 1 : i=j \end{cases}$$

$\boxed{14}$ أوجد منشور لاقير للدالة

$f(x)=e^{x/2}$ حيث $0\le x<\infty$ . الحد الثاني محدد .

$$f(x) = \sum_{n=0}^\infty C_n L_n(x) \qquad \|L_n(x)\|^2 = 1$$

$$C_n = \frac{\langle f(x), L_n(x)\rangle}{\|L_n(x)\|^2} ; \qquad \langle f(x), L_n(x)\rangle = \int_0^\infty f(x)L_n(x)e^{-x}dx$$

$$C_0 = \int_0^\infty e^{x/2}e^{-x}dx = \int_0^\infty e^{-x/2}dx \qquad \left[\frac{e^{-x/2}}{-1/2}\right]_0^\infty = [0-(-2)] = 2$$

$\boxed{C_0=2}$

$$C_1 = \int_0^\infty e^{x/2}(1-x)e^{-x}dx = \int_0^\infty (1-x)e^{-x/2}dx$$

$$u=(1-x) \Rightarrow du=-dx, \quad dv=e^{-x/2}dx \Rightarrow v=-2e^{-x/2}$$

$$\Rightarrow -(1-x)2e^{-x/2}\Big|_0^\infty - 2\int_0^\infty e^{-x/2}dx = 2-2(2) = -2$$

أكمل . لا تنسى الحساب

$\boxed{٢٨}$ تمرين:

$$\int_0^\infty x^n e^{-x}\,dx = n!$$

$$\because \int_0^\infty x e^{-x}\,dx = 1 \quad , \quad \int_0^\infty e^{-x}\,dx = 1$$

$$\int_0^\infty x^2 e^{-x}\,dx = -x^2 e^{-x}\Big|_0^\infty + 2\int_0^\infty x e^{-x}\,dx = \boxed{2}$$

$$\int_0^\infty x^3 e^{-x}\,dx = -x^3 e^{-x}\Big|_0^\infty + 3\int_0^\infty x^2 e^{-x}\,dx = 3\cdot 2 = 3!$$

$$\int_0^\infty x^4 e^{-x}\,dx = x^4 e^{-x}\Big|_0^\infty + 4\int_0^\infty x^3 e^{-x}\,dx = 4\cdot 3\cdot 2\cdot 1 = 4!$$

(وهكذا