

$$(1-x) \frac{d^{n+1}}{dx^{n+1}} x^{n+1} = 2^n n! dx$$

$$(1-x^2) P_n''(x) - 2x P_n'(x) + n(n+1) P_n(x) = 0 \quad (4.2)$$

$$f(x) = x^2 \quad \text{فوجدت } f(x) = x^2 \quad (6)$$

$$P_0(x) = 1, P_1(x) = x, P_2 = \frac{3x^2 - 1}{2}$$

$$P_2 \sim \Rightarrow 2P_2(x) = 3x^2 - 1 \Rightarrow 3x^2 = 2P_2(x) + 1$$

$$\Rightarrow 3x^2 = 2P_2(x) + P_0(x) \Rightarrow x^2 = \frac{1}{3}(2P_2(x) + P_0(x))$$

$$x^2 = \frac{2}{3}P_2(x) + \frac{1}{3}P_0(x)$$

$$f(x) = |x|; \quad -1 \leq x \leq 1 \quad (7)$$

$$f(x) \sim \sum_{n=0}^{\infty} C_n P_n(x), \quad -1 \leq x \leq 1$$

$$C_n = \frac{1}{\|P_n(x)\|^2} \int_{-1}^1 f(x) P_n(x) dx$$

$$C_n = \frac{2n+1}{2} \int_{-1}^1 f(x) P_n(x) dx$$

$$C_0 = \frac{1}{2} \int_{-1}^1 f(x) P_0(x) dx = \frac{1}{2} \left[\int_{-1}^0 -x dx + \int_0^1 x dx \right] = \frac{1}{2} \int_0^1 x dx = \frac{1}{4}$$

$$C_1 = \frac{3}{2} \int_{-1}^1 f(x) P_1(x) dx = \frac{3}{2} \int_{-1}^1 f(x) x dx = \frac{3}{2} \left[\int_{-1}^0 -x^2 dx + \int_0^1 x^2 dx \right] = 0$$

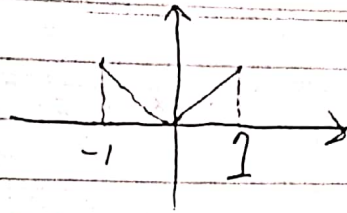
لاحظ
العلاقة
تستخرج من المعادلة
بالفردية غير متساوية
تصبح المتساوية
فكسر المتساوية
K=0
K=1
وهكذا

$$C_2 = \frac{5}{2} \int_{-1}^1 f(x) P_2(x) dx = \frac{5}{2} \cdot 2 \int_0^1 x \cdot \frac{1}{2} (3x^2 - 1) dx = \frac{5}{8}$$

$$C_3 = 0, \quad C_4 = \frac{9}{2} \int_{-1}^1 f(x) P_4(x) dx = \frac{9}{2} \int_0^1 x \dots dx = \frac{-3}{16}$$

$$f(x) \sim C_0 P_0(x) + C_2 P_2(x) + C_4 P_4(x) + \dots$$

$$\sim \frac{1}{2} P_0(x) + \frac{5}{8} P_2(x) - \frac{3}{16} P_4(x) + \dots$$



فردية دالة

$$P_n(1) = 1$$

$$P_n(-1) = (-1)^n$$

$$f(1) = 1, \quad f(1) = \frac{1}{2} + \frac{5}{8} - \frac{3}{16} + \dots$$

$$f(x) = \begin{cases} -1 & -1 < x < 0 \\ 1 & 0 < x < 1 \end{cases}$$

(8)

$$f(x) \sim \sum_{n=0}^{\infty} C_{2n+1} P_{2n+1}(x)$$

$$C_{2n+1} = (4n+3) \int_0^1 f(x) P_{2n+1}(x) dx = (4n+3) \int_0^1 P_{2n+1}(x) dx$$

في التمرين (11.2.2)

$$\Rightarrow \int_x^1 P_n(t) dt = \frac{1}{2n+1} [P_{n+1}(x) - P_n(x)]$$

بدل في هذه العلاقة $x=0$ و $2n+1 > n$ في كل مرة

$$\int_0^1 P_{2n+1}(t) dt = \frac{1}{4n+3} [P_{2n}(0) - P_{2n+2}(0)]$$

$$(4n+3) \int_0^1 P_{2n+1}(t) dt = [P_{2n}(0) - P_{2n+2}(0)]$$

$$\therefore C_{2n+1} = [P_{2n}(0) - P_{2n+2}(0)]$$

$$n=0 \Rightarrow C_1 = [P_0(0) - P_2(0)] = 1 - (-1/2) = 3/2$$

$$n=1 \Rightarrow C_3 = [P_2(0) - P_4(0)] = 1/2 - 3/8 = -1/8$$

$$n=2 \Rightarrow C_5 = [P_4(0) - P_6(0)] = ?$$

$$f(x) \sim \frac{3}{2} P_1(x) - \frac{1}{8} P_3(x) + \square P_5(x) + \dots$$

$$\frac{f(0^-) + f(0^+)}{2} = \frac{7+1}{2} = 4 \text{ في } x=0$$

$$= \frac{3}{2} P_1(0) - \frac{1}{8} P_3(0) + \square P_5(0) = 0$$

④ $(n+1) P_n(x) = P'_{n+1}(x) - x P'_n(x) \quad \forall n \in \mathbb{N}$

نقوم بالتكامل

$$(1-2tx+t^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(x) t^n$$

نقوم بالتكامل

$$-\frac{1}{2}(1-2xt+t^2)^{-3/2}(-2t) = \sum_{n=0}^{\infty} P_n'(x)t^n$$

(1-2xt+t^2) → ضرب الطرفين

$$(1-2xt+t^2)t = \sum_{n=0}^{\infty} P_n'(x)t^{n+1}$$

$$\Rightarrow \sum_{n=0}^{\infty} P_n'(x)t^{n+1} = \sum_{n=0}^{\infty} P_n'(x)t^n - 2x \sum_{n=0}^{\infty} P_n'(x)t^{n+1} + \sum_{n=0}^{\infty} P_n'(x)t^{n+2}$$

$$\Rightarrow tP_0'(x) + \sum_{n=1}^{\infty} P_n'(x)t^n = P_0'(x) + P_1'(x)t$$

$$+ \sum_{n=2}^{\infty} P_n'(x)t^n - 2x P_0'(x)t - 2x \sum_{n=1}^{\infty} P_n'(x)t^{n+1} + \sum_{n=2}^{\infty} P_n'(x)t^{n+2}$$

$$\Rightarrow t P_0'(x) + \sum_{n=1}^{\infty} P_n'(x)t^n = 0 + t + \sum_{n=1}^{\infty} P_{n+1}'(x)t^{n+1}$$

$$- 2x \sum_{n=1}^{\infty} P_n'(x)t^{n+1} + \sum_{n=1}^{\infty} P_{n-1}'(x)t^{n+1}$$

$$\Rightarrow \sum_{n=1}^{\infty} P_n'(x)t^{n+1} = \sum_{n=1}^{\infty} [P_{n+1}'(x) - 2x P_n'(x) + P_{n-1}'(x)] t^{n+1}$$

$$\therefore P_n'(x) - P_{n+1}'(x) - 2x P_n'(x) + P_{n-1}'(x)$$

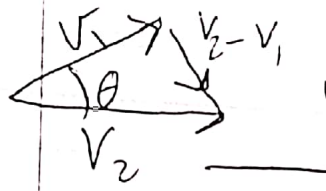
$$P_n'(x) = P_{n+1}'(x) - x P_n'(x) - x P_n'(x) + P_{n-1}'(x)$$

$$= P_{n+1}'(x) - x P_n'(x) - n P_n'(x)$$

$$\Rightarrow (1+n)P_n'(x) = P_{n+1}'(x) - x P_n'(x)$$

الطرفين

0v



(1)

الزاوية بين v_1, v_2 هي θ

$$\frac{1}{\sqrt{1 - 2x + x^2}} = \sum_{n=0}^{\infty} P_n(x) x^n$$

$$\frac{1}{|re^{i\theta} - 1|} = \frac{1}{\sqrt{r^2 - 2r\cos\theta + 1}} = \sum_{n=0}^{\infty} P_n(\cos\theta) r^n$$

حيث $r < 1$

$$re^{i\theta} - 1 = r(\cos\theta + i\sin\theta) - 1 = (r\cos\theta - 1) + i(r\sin\theta)$$

$$|re^{i\theta} - 1| = \sqrt{(r\cos\theta - 1)^2 + (r\sin\theta)^2}$$

$$= \sqrt{r^2 \cos^2\theta - 2r\cos\theta + 1 + r^2 \sin^2\theta}$$

$$= \sqrt{r^2 - 2r\cos\theta + 1}$$

$$\Rightarrow \frac{1}{|re^{i\theta} - 1|} = \frac{1}{\sqrt{r^2 - 2r\cos\theta + 1}} = \sum_{n=0}^{\infty} P_n(\cos\theta) r^n$$

(وذلك بافتراض ان $t = r$ و $x = \cos\theta$ في دالة التوليد (1))

$$\frac{1}{\|v_2 - v_1\|} = \frac{1}{\|v_2\|} \sum_{n=0}^{\infty} P_n(\cos\theta) \left(\frac{\|v_1\|}{\|v_2\|}\right)^n$$

لدي متجهين v_1, v_2 في المستوى بينهما الزاوية θ حيث $\|v_1\| < \|v_2\|$

(on)

○ r

$$\|V_1 - V_2\|^2 = \|V_1\|^2 - 2\|V_1\|\|V_2\|\cos\theta + \|V_2\|^2$$

$$\left\| \frac{V_1}{V_2} \right\| = r < 1 \Leftrightarrow \|V_1\| < \|V_2\| \quad \dots$$

$$\|V_1\| = r \|V_2\| \quad \Leftrightarrow$$

$$\|V_2 - V_1\|^2 = \|V_2\|^2 + r^2 \|V_2\|^2 - 2\|V_2\|^2 r \cos\theta$$

$$= \|V_2\|^2 (1 + r^2 - 2r \cos\theta)$$

$$\Rightarrow \|V_2 - V_1\| = \|V_2\| \sqrt{r^2 - 2r \cos\theta + 1}$$

$$\Rightarrow \frac{1}{\|V_2 - V_1\|} = \frac{1}{\|V_2\| \sqrt{r^2 - 2r \cos\theta + 1}} = \frac{1}{\|V_2\|} \frac{1}{\sqrt{r^2 - 2r \cos\theta + 1}}$$

$$= \frac{1}{\|V_2\|} \sum_{n=0}^{\infty} P_n(\cos\theta) r^n = \frac{1}{\|V_2\|} \sum_{n=0}^{\infty} P_n(\cos\theta) \left\| \frac{V_1}{V_2} \right\|^n$$

2) $P'_{n+1}(x) - P'_n(x) = (2n+1)P_n(x) ; \forall n \in \mathbb{N}$

الشيء: مع صيغة رودريغز $P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2-1)^n$ (*)

نستعمل الطريقة $P'_n(x) = \frac{2n}{2^n n!} \frac{d^n}{dx^n} [x(x^2-1)^{n-1}]$

$$= \frac{2n}{2^n n!} \frac{d^{n-1}}{dx^{n-1}} [(2n-2)x^2(x^2-1)^{n-2} + (x^2-1)(x^2-1)^{n-2}]$$

$$P'_n(x) = \frac{1}{2^{n-1} (n-1)!} \frac{d^{n-1}}{dx^{n-1}} [(2n-1)x^2-1)(x^2-1)^{n-2}]$$

$\Leftrightarrow n+1 \geq n$ \int

$$P'_{n+1}(x) = \frac{1}{2^{n+1}n!} \frac{d^n}{dx^n} \left[((2n+1)x^2 - 1)(x^2 - 1)^{n-1} \right] \quad \text{--- ①}$$

$$P'_{n-1}(x) = \frac{1}{2^n(n-1)!} \frac{d^{n-1}}{dx^{n-1}} \left[(x^2 - 1)^{n-1} \right]$$

↔ (*) ↻

مشتق

$$P'_{n-1}(x) = \frac{2n}{2^n n!} \frac{d^n}{dx^n} \left[(x^2 - 1)^{n-1} \right] \quad \text{②}$$

$$\text{①} - \text{②} \Rightarrow P'_{n+1}(x) - P'_{n-1}(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} \left[((2n+1)x^2 - 1)(x^2 - 1)^{n-1} - 2n(x^2 - 1)^{n-1} \right]$$

$$= \frac{1}{2^n n!} \frac{d^n}{dx^n} \left[(2n+1)x^2(x^2 - 1)^{n-1} - (2n+1)(x^2 - 1)^{n-1} \right]$$

$$= \frac{(2n+1)}{2^n n!} \frac{d^n}{dx^n} \left[(x^2 - 1)(x^2 - 1)^{n-1} \right] = \frac{2n+1}{2^n n!} \frac{d^n}{dx^n} (x^2 - 1)^n$$

$$= (2n+1) P_n(x)$$

$$\int_1^x P_n(x) dx = \frac{1}{2n+1} \left[\int_1^x P'_{n+1}(t) dt - \int_1^x P'_{n-1}(t) dt \right]$$

$$= \frac{1}{2n+1} \left[P_{n+1}(x) \Big|_1^x - P_{n-1}(x) \Big|_1^x \right]$$

$$= \frac{1}{2n+1} \left[P_{n+1}(x) - P_{n-1}(x) - P_{n+1}(1) + P_{n-1}(1) \right]$$

$$= \frac{1}{2n+1} \left[P_{n+1}(x) - P_{n-1}(x) - 1 + 1 \right]$$

$$= \frac{1}{2n+1} \left[P_{n+1}(x) - P_{n-1}(x) \right]$$

$\frac{d}{dx}$

$$\boxed{3} \quad (2n+1)x P_n(x) = (n+1)P_{n+1}(x) + nP_{n-1}(x)$$

من لالة التوليد $= C_2(x)$

$$[1-2xt+t^2]^{-1/2} = \sum_{n=0}^{\infty} P_n(x) t^n$$

نفسه بالاشتراك t

$$-\frac{1}{2}(-2x+2t)[1-2xt+t^2]^{-3/2} = \sum_{n=1}^{\infty} n P_n(x) t^{n-1}$$

$$\Rightarrow (x-t)[1-2xt+t^2]^{-3/2} = \sum_{n=1}^{\infty} n P_n(x) t^{n-1}$$

في الالة $(1-2xt+t^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(x) t^n$ وجعل t بالاطراف الاخرى فنحصل على $(1-2xt+t^2)$

$$\sum_{n=1}^{\infty} (x-t) P_n(x) t^{n-1} = \sum_{n=1}^{\infty} (1-2xt+t^2) n P_n(x) t^{n-1}$$

$$\Rightarrow \sum_{n=0}^{\infty} x P_n(x) t^n - \sum_{n=0}^{\infty} P_n(x) t^{n+1} = \sum_{n=1}^{\infty} n P_n(x) t^n - \sum_{n=1}^{\infty} 2nx P_n(x) t^n + \sum_{n=1}^{\infty} n P_n(x) t^{n+1}$$

نوع الايسر t في

$$\Rightarrow \sum_{n=0}^{\infty} x P_n(x) t^n - \sum_{n=1}^{\infty} P_{n-1}(x) t^n = \sum_{n=0}^{\infty} (n+1) P_{n+1}(x) t^n - \sum_{n=1}^{\infty} 2nx P_n(x) t^n + \sum_{n=2}^{\infty} (n-1) P_{n-1}(x) t^n$$

نوع الايسر x

$$\Rightarrow x P_0(x) + x P_1(x) + \sum_{n=2}^{\infty} x P_n(x) t^n - P_0(x) t - \sum_{n=2}^{\infty} P_{n-1}(x) t^n$$

$$= P_1(x) + 2P_2(x)t + \sum_{n=2}^{\infty} (n+1) P_{n+1}(x) t^n - 2x P_1(x) t - \sum_{n=2}^{\infty} 2nx P_n(x) t^n + \sum_{n=2}^{\infty} (n-1) P_{n-1}(x) t^n$$

$$\sum_{n=2}^{\infty} [x P_n(x) - P_{n+1}(x)] t^n = \sum_{n=2}^{\infty} [(n+1) P_{n+1}(x) - 2nx P_n(x) + (n-1) P_{n-1}(x)] t^n$$

$$x P_n(x) - P_{n+1}(x) = (n+1) P_{n+1}(x) - 2nx P_n(x) + (n-1) P_{n-1}(x)$$

فان وانه صحيح على الساع

(4) $(n+1) P_n(x) = P_{n+1}'(x) - x P_n'(x)$

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2-1)^n$$

$$\Rightarrow P_n'(x) = \frac{2n}{2^n n!} \frac{d^n}{dx^n} [x(x^2-1)^{n-1}] = \frac{1}{2^{n-1} (n-1)!} \frac{d^n}{dx^n} [x(x^2-1)^{n-1}]$$

$$\Rightarrow P_{n+1}'(x) = \frac{1}{2^n n!} \frac{d^{n+1}}{dx^{n+1}} [x(x^2-1)^n]$$

نتیجه از این است:

$$\frac{d^{n+1}}{dx^{n+1}} [x f(x)] = x \frac{d^{n+1}}{dx^{n+1}} f(x) + (n+1) \frac{d^n}{dx^n} f(x)$$

$$\Rightarrow P_{n+1}'(x) = \frac{1}{2^n n!} \left[x \frac{d^{n+1}}{dx^{n+1}} (x^2-1)^n + (n+1) \frac{d^n}{dx^n} (x^2-1)^n \right]$$

$$\frac{d^{n+1}}{dx^{n+1}} [x(x^2-1)^n] = x \frac{1}{2^n n!} \frac{d^{n+1}}{dx^{n+1}} (x^2-1)^n + (n+1) \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2-1)^n$$

$$= x \frac{d}{dx} \left[\frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2-1)^n \right]$$

$$\Rightarrow P_{n+1}'(x) = x P_n'(x) + (n+1) P_n(x)$$

$$(n+1) P_n(x) = P_{n+1}'(x) - x P_n'(x)$$

(5)

از 5، 4

$$(2) - (4) = (5)$$

(4)

$$P_n(x) = \sum_{k=0}^{n/2} (-1)^k \frac{(2n-2k)!}{2^n k! (n-k)! (n-2k)!} x^{n-2k}$$

$$n = 2m + 1$$

$$P_{2m+1}(x) = \binom{2m+1}{0} x^{2m+1} + \dots + \binom{2m+1}{2} x^2$$

$$P_{2m}(x) = \binom{2m}{0} x^{2m} + \dots + \binom{2m}{2} x^2 + (-1)^m \frac{(2m)!}{2^m m! m!}$$

$$\frac{(-1)^m (2m)!}{2^{2m} (m!)^2} = (-1)^m \frac{1 \cdot 3 \cdot 5 \dots (2m-1)}{2 \cdot 4 \cdot 6 \dots (2m)}$$

$$\frac{m! (m+1)(m+2) \dots (2m)}{2^m 2 \cdot 4 \cdot 6 \dots (2m)}$$

الإمام استخدم الصيغة السابقة على

$$\frac{2!}{2^2 1!} = \frac{2}{4} = \frac{1}{2} \quad m=1$$

المساوية صحيحة! $\frac{1}{2}$ النتيجة

لنفتقنا إلى استراة جديدة لـ m نفترض

$$\frac{(2m+2)!}{2^{2m+2} (m+1)! (m+1)!} = \frac{1 \cdot 3 \cdot 5 \dots (2m+1)}{2 \cdot 4 \cdot 6 \dots (2m+2)}$$

$$\frac{(2m)! (2m+1)(2m+2)}{2^{2m} (m!)^2 2^2 (m+1)^2} = \frac{1 \cdot 3 \cdot 5 \dots (2m-1)(2m+1)(2m+2)}{2 \cdot 4 \cdot 6 \dots (2m)(2m+2)^2}$$

$(m+1)! = m!(m+1)$

$$= \frac{1 \cdot 3 \cdot 5 \dots (2m-1) (2m+1)}{2 \cdot 4 \cdot 6 \dots (2m) (2m+2)}$$