$$(1-x^2)\,\frac{\quad}{2^n n!}\;\frac{\quad}{dx^2}\qquad 2^n n!\; dx$$

$$(1-x^2)P_n''(x)-2xP_n'(x)+(n+1)nP_n(x)=0$$

$$(4.2)$$

ملاحظة
العلاقة ⑦
تستخدم التي فيها
بالضرب عند حساب
ثم لاشتقاقها
يكون لدينا الى
$K=0$ وثم نشتق
يكون $K=1$
وهكذا .

6 مثل f بسلسلة لوجاندر

$$f(x)=x^2$$

$$P_0(x)=1,\; P_1(x)=x,\; P_2=\frac{3x^2-1}{2}$$

$$P_2 \Rightarrow 2P_2(x)=3x^2-1 \Rightarrow 3x^2=2P_2(x)+1$$

$$\Rightarrow 3x^2=2P_2(x)+P_0(x) \Rightarrow x^2=\frac{1}{3}\left(2P_2(x)+P_0(x)\right)$$

$$x^2=\frac{2}{3}P_2(x)+\frac{1}{3}P_0(x)$$

7

$$f(x)=|x|\;;\quad -1\le x\le 1$$

$$f(x)\sim\sum_{n=0}^{\infty}C_n P_n(x),\; -1\le x\le 1$$

$$C_n=\frac{1}{\|P_n(x)\|^2}\int_{-1}^{1}f(x)P_n(x)\,dx$$

$$C_n=\frac{2n+1}{2}\int_{-1}^{1}f(x)P_n(x)\,dx$$

زوجية

$$C_0=\frac{1}{2}\int_{-1}^{1}f(x)\underbrace{P_0(x)}_{=1}\,dx=\frac{1}{2}\left[\int_{-1}^{0}-x\,dx+\int_{0}^{1}x\,dx\right]=\frac{2}{2}\int_0^1 x\,dx=\boxed{1/2}$$

$$C_1=\frac{3}{2}\int_{-1}^{1}f(x)P_1(x)\,dx=\frac{3}{2}\int_{-1}^{1}f(x)\,x\,dx=\frac{3}{2}\left[\int_{-1}^{0}-x^2\,dx+\int_{0}^{1}x^2\,dx\right]$$

$$=\boxed{0}$$

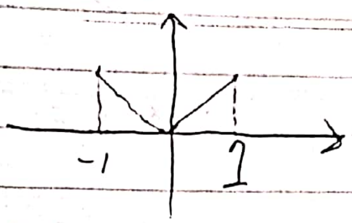
$$C_2 = \frac{5}{2}\int_{-1}^{1} f(x)\, P_2(x)\, dx = \frac{5}{2}\cdot 2\int_0^1 x\,\frac{1}{2}\,(3x^2-1)\,dx = \frac{5}{8}$$

$$C_3 = 0\,, \quad C_4 = \frac{9}{2}\int_{-1}^{1} f(x)\, P_4(x)\, dx = \frac{9}{2}\, 2\int_0^1 x \cdots dx = \boxed{\frac{-3}{16}}$$

...

$$f(x) \sim C_0 P_0(x) + C_2 P_2(x) + C_4 P_4(x) + \cdots$$

$$\sim \frac{1}{2} P_0(x) + \frac{5}{8} P_2(x) - \frac{3}{16} P_4(x) + \cdots$$



ملاحظة هامة جداً ,

$$P_n(1) = 1$$

$$P_n(-1) = (-1)^n$$

الطرف الأيسر $= f(1) = 1$, الطرف الأيمن $= \frac{1}{2} + \frac{5}{8} - \frac{3}{16} + \cdots$

$\frac{15}{16} + \cdots$ أي أن

(8)

$$f(x) = \begin{cases} -1 & -1 < x < 0 \\ 1 & 0 < x < 1 \end{cases}$$

f دالة فردية

$$f(x) \sim \sum_{n=0}^{\infty} C_{2n+1}\, P_{2n+1}(x)$$

$$C_{2n+1} = (4n+3)\int_0^1 f(x)\, P_{2n+1}(x)\, dx$$

$$= (4n+3)\int_0^1 P_{2n+1}(x)\, dx$$

من التمرين (1.2.2)

$$\Rightarrow \int_x^1 P_n(t)\,dt = \frac{1}{2n+1}\left[P_{n-1}(x) - P_{n+1}(x)\right]$$

بدل في هذه العلاقة مكان كل n بـ $2n+1$ وبدل كل x بـ 0 .

$$\int_0^1 P_{2n+1}(t)\,dt = \frac{1}{4n+3}\left[P_{2n}(0) - P_{2n+2}(0)\right]$$

$$(4n+3)\int_0^1 P_{2n+1}(t)\,dt = \left[P_{2n}(0) - P_{2n+2}(0)\right]$$

$$\therefore C_{2n+1} = \left[P_{2n}(0) - P_{2n+2}(0)\right]$$

$$n=0 \Rightarrow C_1 = \left[P_0(0) - P_2(0)\right] = 1 - \left(-\tfrac{1}{2}\right) = \tfrac{3}{2}$$

$$n=1 \Rightarrow C_3 = \left[P_2(0) - P_4(0)\right] = -\tfrac{1}{2} - \tfrac{3}{8} = \tfrac{-7}{8}$$

$$n=2 \Rightarrow C_5 = \left[P_4 - P_6(0)\right] = ?$$ احسب

$\vdots$

$$f(x) \sim \frac{3}{2}P_1(x) - \frac{7}{8}P_3(x) + \square P_5(x) + \cdots$$

$$\frac{f(0^-)+f(0^+)}{2} = \frac{-1+1}{2} = 0$$ عند $x=0$ بتعويض

$$= \underbrace{\frac{3}{2}P_1(0) - \frac{7}{8}P_3(0) + \square P_5(0)}_{\text{الطرف الأيمن (السلسلة)}} = 0$$

④ $(n+1)P_n(x) = P'_{n+1}(x) - xP'_n(x);\ \forall n \in \mathbb{N}$

الاثبات من دالة التوليد

$$(1-2tx+t^2)^{-1/2} = \sum_0^\infty P_n(x)\,t^n$$

نشتق الطرفين بالنسبة لـ x .

07

بـ س

$$-\tfrac{1}{2}(1-2xt+t^2)^{-3/2}(-2t) = \sum^{\infty} P_n'(x)\, t^n$$

نضرب الطرفين بـ $(1-2xt+t^2)$

$$(1-2xt+t^2)\,t = \sum_0^{\infty} P_n'(x)\, t^n (1-2xt+t^2)$$

$$\Rightarrow \sum_{n=0}^{\infty} P_n(x)\, t^{n+1} = \sum_{n=0}^{\infty} P_n'(x)\, t^n - 2x \sum_0^{\infty} P_n'(x)\, t^{n+1}$$

$$+ \sum_0^{\infty} P_n'(x)\, t^{n+2}$$

$$\Rightarrow t P_0(x) + \sum_1^{\infty} P_n(x)\, t^{n+1} = P_0'(x) + P_1'(x)\, t$$

$$+ \sum_{n=2}^{\infty} P_n'(x)\, \boxed{t^n} - 2x P_0'(x)\, t - 2x \sum_{n=1}^{\infty} P_n'(x)\, t^{n+1}$$

$$+ \sum_0^{\infty} P_n'(x)\, t^{n+2}$$

$$\Rightarrow t + \sum_0^{\infty} P_n(x)\, t^{n+1} = 0 + t + \sum_1 P_{n+1}'\, \boxed{t^{n+1}}$$

$$-2x \sum_1^{\infty} P_n'(x)\, t^{n+1} + \sum_{n=1} P_{n-1}'(x)\, t^{n+1}$$

$$\Rightarrow \sum_1^{\infty} P_n(x)\, t^{n+1} = \sum \left[P_{n+1}'(x) - 2x P_n'(x) + P_{n-1}'(x) \right] t^{n+1}$$

$$\therefore P_n(x) = P_{n+1}'(x) - 2x P_n'(x) + P_{n-1}'(x)$$

$$P_n(x) = P_{n+1}'(x) - x P_n'(x) - \underbrace{x P_n'(x) + P_{n-1}'(x)}_{\text{معلوم}}$$

$$= P_{n+1}'(x) - x P_n'(x) - n P_n(x)$$

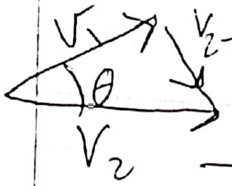
$$\Rightarrow (1+n) P_n(x) = P_{n+1}'(x) - x P_n'(x)$$

٥٧

بسم الله

٢/٦ ربيع ثاني

الدالة المولدة ①



$$\frac{1}{\sqrt{t^2-2xt+1}} = \sum_{n=0}^{\infty} P_n(x)\, t^n$$

$\boxed{1}$

$$\frac{1}{|re^{i\theta}-1|} = \frac{1}{\sqrt{r^2-2r\cos\theta+1}} = \sum_{n=0}^{\infty} P_n(\cos\theta)\, r^n$$

حيث $|r|<1$

الاثبات:

$$re^{i\theta}-1 = r(\cos\theta + i\sin\theta)-1$$

$$= (r\cos\theta-1) + i(r\sin\theta)$$

$$|re^{i\theta}-1| = \sqrt{(r\cos\theta-1)^2+(r\sin\theta)^2}$$

$$= \sqrt{r^2\cos^2\theta - 2r\cos\theta + 1 + r^2\sin^2\theta}$$

$$= \sqrt{r^2-2r\cos\theta+1}$$

$$\Rightarrow \frac{1}{|re^{i\theta}-1|} = \frac{1}{\sqrt{r^2-2r\cos\theta+1}} = \sum_{n=0}^{\infty} P_n(\cos\theta)\, r^n$$

(وذلك باستبدال كل t بـ r وكل x بـ $\cos\theta$ في دالة التوليد ①)

* استنتج أن

$$\frac{1}{\|V_2-V_1\|} = \frac{1}{\|V_2\|} \sum_{n=0}^{\infty} P_n(\cos\theta) \left\|\frac{V_1}{V_2}\right\|^n$$

لأي متجهين V_1, V_2 في المستوى بينهما الزاوية θ حيث $\left\|\frac{V_1}{V_2}\right\|<1$

ON

حل ٤.

$\|v_1-v_2\|^2=\|v_1\|^2-2\|v_1\|\|v_2\|\cos\theta+\ldots$

$\left\|\frac{V_1}{V_2}\right\|=r<1 \Leftarrow \left\|\frac{V_1}{V_2}\right\|<1$ ∴

$\|V_1\|=|r|\,\|V_2\|$ $\Leftarrow$

$\|V_2-V_1\|^2=\|V_2\|^2+r^2\|V_2\|^2-2|r|\|V_2\|^2\cos\theta$

$=\|V_2\|^2\left(1+r^2-2|r|\cos\theta\right)$

$\Rightarrow \|V_2-V_1\|=\|V_2\|\sqrt{r^2-2|r|\cos\theta+1}$

$\Rightarrow \frac{1}{\|V_2-V_1\|}=\frac{1}{\|V_2\|\sqrt{r^2-2r\cos\theta+1}}=\frac{1}{\|V_2\|}\frac{1}{\sqrt{r^2-2r\cos\theta+1}}=\phi$

$=\frac{1}{\|V_2\|}\sum_{n=0}^{\infty}P_n(\cos\theta)r^n=\frac{1}{\|V_2\|}\sum_{n=0}^{\infty}P_n(\cos\theta)\left\|\frac{V_1}{V_2}\right\|^n$

2) $P'_{n+1}(x)-P'_{n-1}(x)=(2n+1)P_n(x)\ ;\ \forall n\in\mathbb{N}$

الاثبات: من صيغة رودريغز

$P_n(x)=\frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n$ (*)

نشتق الطرفين $\Leftarrow$

$P'(x)=\frac{2n}{2^n n!}\frac{d^n}{dx^n}\left[x(x^2-1)^{n-1}\right]$

$=\frac{2n}{2^n n!}\frac{d^{n-1}}{dx^{n-1}}\left[(2n-2)x^2(x^2-1)^{n-2}+(x^2-1)(x^2-1)^{n-2}\right]$

$P'_n(x)=\frac{1}{2^{n-1}(n-1)!}\frac{d^{n-1}}{dx^{n-1}}\left[\left((2n-1)x^2-1\right)(x^2-1)^{n-2}\right]$

استبدال n ب $n+1$ $\Leftarrow$

09

$$P'_{n+1}(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n}\left[\left((2n+1)x^2-1\right)(x^2-1)^{n-1}\right] \quad — ①$$

$$\Leftarrow \circledast$$

$$P'_{n-1}(x) = \frac{1}{2^{n-1}(n-1)!} \frac{d^{n-1}}{dx^{n-1}}\left[(x^2-1)^{n-1}\right]$$

با جایگذاری

$$P'_{n-1}(x) = \frac{2n}{2^n n!} \frac{d^n}{dx^n}\left[(x^2-1)^{n-1}\right] \quad ②$$

$$① - ② \Rightarrow P'_{n+1}(x) - P'_{n-1}(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n}\left[\left((2n+1)x^2-1\right)(x^2-1)^{n-1} - 2n(x^2-1)^{n-1}\right]$$

$$= \frac{1}{2^n n!} \frac{d^n}{dx^n}\left[(2n+1)x^2(x^2-1)^{n-1} - (2n+1)(x^2-1)^{n-1}\right]$$

$$= \frac{(2n+1)}{2^n n!} \frac{d^n}{dx^n}\left[(x^2-1)(x^2-1)^{n-1}\right] = \frac{2n+1}{2^n n!} \frac{d^n}{dx^n}(x^2-1)^n$$

$$= (2n+1) P_n(x)$$

$$\circledast \quad \int_1^x P_n(t)\, dt = \frac{1}{2n+1}\left[\int_1^x P'_{n+1}(t)\, dt - \int_1^x P'_{n-1}(t)\, dt\right]$$

$$= \frac{1}{2n+1}\left[P_{n+1}(t)\Big|_1^x - P_{n-1}(t)\Big|_1^x\right]$$

$$= \frac{1}{2n+1}\left[P_{n+1}(x) - P_{n-1}(x) - P_{n+1}(1) + P_{n-1}(1)\right]$$

$$= \frac{1}{2n+1}\left[P_{n+1}(x) - P_{n-1}(x) - 1 + 1\right]$$

$$= \frac{1}{2n+1}\left[P_{n+1}(x) - P_{n-1}(x)\right]$$

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بسم الله

$\boxed{3}$ $(2n+1)\, x P_n(x) = (n+1) P_{n+1}(x) + n P_{n-1}(x)$

نثبت من الدالة المولدة

$$[1-2xt+t^2]^{-1/2} = \sum_0^\infty P_n(x)\, t^n$$

نفاضل بالنسبة لـ t

$$-\frac{1}{2}(-2x+2t)\left[1-2xt+t^2\right]^{-3/2} = \sum_1^\infty n P_n(x)\, t^{n-1}$$

$$\Rightarrow (x-t)\left[1-2xt+t^2\right]^{-3/2} = \sum_1^\infty n P_n(x)\, t^{n-1}$$

نضرب الطرفين $(1-2xt+t^2)$ ونعمل تعويض $(1-2xt+t^2)^{-1/2} = \sum P_n(x) t^n$

$$\sum_1^\infty (x-t) P_n(x)\, t^n = \sum_1^\infty (1-2xt+t^2)\, n P_n(x)\, t^{n-1}$$

$$\Rightarrow \sum_0^\infty x P_n(x)\, t^n - \sum_0^\infty P_n(x)\, t^{n+1} = \sum_{n=1}^\infty n P_n(x)\, t^{n-1} - \sum_1^\infty 2nx P_n(x)\, t^n + \sum_1^\infty n P_n(x)\, t^{n+1}$$

نوحد الاسس على t

$$\Rightarrow \sum_{n=0}^\infty x P_n(x)\, t^n - \sum_1^\infty P_{n-1}(x)\, t^n = \sum_{n=0}^\infty (n+1) P_{n+1}(x)\, t^n - \sum_1^\infty 2nx P_n(x)\, t^n$$

$$+ \sum_2^\infty (n-1) P_{n-1}(x)\, t^n$$

$$\Rightarrow \underbrace{x P_0(x)}_{x} + x t P_1(x) + \sum_2^\infty x P_n(x)\, t^n - P_0(x)\, t - \sum_2^\infty P_{n-1}(x)\, t^n$$

نأخذ من الاعتبار

$$= \underbrace{P_1(x)}_{x} + 2P_2(x)\, t + \sum_{n=2}^\infty (n+1) P_{n+1}(x)\, t^n - 2x P_1(x)\, t - \sum_2^\infty 2nx P_n(x)\, t^n$$

$$+ \sum_2^\infty (n-1) P_{n-1}(x)\, t^n$$

$$\sum_2^\infty \left[x P_n(x) - P_{n-1}(x)\right] t^n = \sum_2^\infty \left[(n+1) P_{n+1} - 2nx P_n(x) + (n-1) P_{n-1}(x)\right] t^n$$

$$x P_n(x) - P_{n-1}(x) = (n+1) P_{n+1}(x) - 2nx\, P_n(x) + (n-1) P_{n-1}(x)$$

فك واختصر تحصل على الناتج.

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(1)

(1) $(n+1)\,P_n(x) = P'_{n+1}(x) - x\,P'_n(x)$

الاثبات من صيغة رودريغس

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n$$

$$\Rightarrow P'_n(x) = \frac{2n}{2^n n!}\frac{d^n}{dx^n}\left[x(x^2-1)^{n-1}\right] = \frac{1}{2^{n-1}(n-1)!}\frac{d^n}{dx^n}\left[x(x^2-1)^{n-1}\right]$$

$$\Rightarrow P'_{n+1}(x) = \frac{1}{2^n n!}\frac{d^{n+1}}{dx^{n+1}}\left[x(x^2-1)^n\right]$$

نستخدم هذه الصيغة :

$$\frac{d^{n+1}}{dx^{n+1}}\left[x f(x)\right] = x\frac{d^{n+1}}{dx^{n+1}}f(x) + (n+1)\frac{d^n}{dx^n}f(x)$$

$$\Rightarrow P'_{n+1}(x) = \frac{1}{2^n n!}\left[x\frac{d^{n+1}}{dx^{n+1}}(x^2-1)^n + (n+1)\frac{d^n}{dx^n}(x^2-1)^n\right]$$

$$\cancel{\frac{d^n}{dx^n}(x^2-1)^n} = x\frac{1}{2^n n!}\frac{d^{n+1}}{dx^{n+1}}(x^2-1)^n + (n+1)\frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n$$

$$= x\frac{d}{dx}\left[\frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n\right]$$

$$\Rightarrow P'_{n+1}(x) = x P'_n(x) + (n+1)P_n(x)$$

$$(n+1)P_n(x) = P'_{n+1}(x) - x P'_n(x)$$

(5)

عبارة عن

(2) − (4) = (5)

(4.1)

(4) $$P_n(x) = \sum_{k=0}^{[n/2]}(-1)^k\frac{(2n-2k)!}{2^n k!(n-k)!(n-2k)!}x^{n-2k}$$

$n = 2m+1$

$$P_{2m+1}(x) = (\ \ )x^{2m+1} + \cdots + (\ \ )x^{1}$$

$n = 2m$

$$P_{2m}(x) = (\ \ )x^{2m} + \cdots + (\ \ )x^{2} + (-1)^{m}\frac{(2m)!}{2^{2m}\, m!\, m!} \qquad k = \left[\frac{n}{2}\right] = m$$

$$(-1)^{m}\frac{(2m)!}{2^{2m}(m!)^2} = (-1)^{m}\frac{1\cdot 3\cdot 5 \cdots (2m-1)}{2\cdot 4\cdot 6 \cdots (2m)}$$

$$\frac{m!\,(m+1)(m+2)\cdots(2m)}{2^{m}\; 2\cdot 4\cdot 6\cdots(2m)}$$

البرهان: نستخدم الاستقراء الرياضي على m

$m=1$:

الأيسر $\frac{2!}{2^2\, 1!\, 1!} = \frac{2}{4} = \frac{1}{2}$

الأيمن $\frac{1}{2}$ إذن صحيحة عند $m=1$

نفرض أن المساواة صحيحة لـ m ثم نبرهن لـ $m+1$

$$\frac{(2m+2)!}{2^{2m+2}(m+1)!\,(m+1)!} \overset{?}{=} \frac{1\cdot 3\cdot 5 \cdots (2m+1)}{2\cdot 4\cdot 6\cdots(2m+2)}$$

$$\frac{(2m)!\,(2m+1)(2m+2)}{2^{2m}(m!)^2\, 2^2\,(m+1)^2} = \frac{1\cdot 3\cdot 5 \cdots (2m-1)(2m+1)(2m+2)}{2\cdot 4\cdot 6\cdots(2m)(2m+2)^2}$$

$(m+1)! = m!(m+1)$ نعوض

$$= \frac{1\cdot 3\cdot 5 \cdots (2m-1)\,\overset{[2(m+1)-1]}{(2m+1)}}{2\cdot 4\cdot 6 \cdots (2m)\,\underset{2(m+1)}{(2m+2)}}$$