$$p(x)u'' + p'(x)u' + r(x)u + \lambda w(x)u = 0 \quad (*)$$

(1)

(i) $x^2 u'' + \lambda u = 0, \ x > 0$

$u'' + \lambda \frac{1}{x^2} u = 0, \ x > 0$

بالمقارنة مع الصيغة العامة في $(*) \Leftarrow$

$p(x) = 1, \ p'(x) = 0, \ r(x) = 0, \ w(x) = \frac{1}{x^2}$

(ii) $\sin x \, u'' + \cos x \, u' + \lambda \sin x \, u = 0, \quad 0 < x < \pi$

بالمقارنة مع $(*) \Leftarrow$

$p(x) = \sin x, \ p'(x) = \cos x, \ r(x) = 0, \ w(x) = \sin x$

(iii) $u'' - u' + \lambda u = 0$

بالضرب في دالة معينة ولتكن s

$su'' - su' + \lambda s u = 0$

$\Rightarrow p(x) = s, \ p' = -s$

$\Rightarrow s' = -s \Rightarrow \frac{s'}{s} = -1 \Rightarrow \ln s = -x \Rightarrow s = e^{-x}$

أو بالضرب في العامل

$s(x) = \frac{1}{p} \exp \int (a_1/p)$

$\therefore s(x) = 1 \exp\left(\int -1 \, dx\right) = e^{-x}$

$e^{-x} u'' - e^{-x} u' + \lambda e^{-x} u = 0$

$p(x) = e^{-x}, \ p'(x) = -e^{-x}; \ r(x) = 0, \ w(x) = e^{-x}$

(iv) $u'' - x^2 u' + \lambda u = 0$

$s(x) = \frac{1}{1} e^{\int -x^2 dx} = e^{-\frac{x^3}{3}}$

Scanned by CamScanner

$e^{-x^3/3} - x^2 e^{-x^3/3} u' + \lambda u e^{-x^3/3} = 0$

$p(x) = e^{-x^3/3}$, $p'(x) = -x^2 e^{-x^3/3}$, $r(x) = 0$, $w(x) = e^{-x^3/3}$

(2)

$\therefore \; p(f'g - fg')\Big|_a^b = 0$

(i) $p(x) = 1$, $a \le x \le b$, $u(a) = u(b)$, $u'(a) = u'(b)$

$1[f'(b)g(b) - f(b)g'(b)] - 1[f'(a)g(a) - f(a)g'(a)]$

$\Rightarrow [f'(a)g(a) - f(a)g'(a)] - [f'(a)g(a) - f(a)g'(a)] = 0$

(+) (iii) $p(x) = \sin x$, $0 \le x \le \pi/2$, $u(0) = 1$, $u(\pi/2) = 0$

$\sin\pi/2 \left[f'(\pi/2)g(\pi/2) - f(\pi/2)g'(\pi/2)\right] - \underbrace{\sin(0)}_{\to 0} \left[f'(0)g(0) - g'(0)f(0)\right]$

$1 \cdot [(0)f' - (0)g] = 0$

(+) (ii) $p(x) = x$, $0 < a \le x \le b$, $u(a) = u'(b) = 0$

$b[f'(b)g(b) - f(b)g'(b)] - a[f'(a)g(a) - f(a)g'(a)]$

$\Rightarrow b[(0)g(b) - f(b)(0)] - a[f'(a)(0) - (0)g'(a)] = 0$

(iv) $p(x) = e^{-x}$ $0 < x < 1$

$u(0) = u(1)$, $u'(0) = u'(1)$

$e^{-1}[f'(1)g(1) - f(1)g'(1)] - e^{0}[f'(0)g(0) - g'(0)f(0)]$

$\frac{1}{e}[f'(0)g(0) - f(0)g'(0)] - [f'(0)g(0) - g'(0)f(0)]$

(٨) بسم الله

$$\left[\frac{1}{e}-1\right]\left[f'(0)g(0)-f(0)g'(0)\right]\neq 0$$

(V) $p(x)=x^2,\ 0<x<b,\ u'(0)=u(b),\ u'(b)=u(b)$

$$x^2\left[f'g-g'f\right]\Big|_a^b=b^2\left[f'(b)g(b)-g'(b)f'(b)\right]-0^2\left[f'(0)g(0)-g'(0)f(0)\right]$$

$$=b^2\left[f(b)g(b)-g(b)f(b)\right]-0^2\left[f(b)g(0)-g(b)f(0)\right]$$

$$=b^2[0]=0$$

(Vi) $p(x)=x^2,\ 0<x<b$

$u'(0)=u(b),\ u'(b)=u(0)$

$$b^2\left[f'(b)g(b)-g'(b)f(b)\right]-0\left[f'(0)g(0)-f(0)g'(0)\right]$$

$$=b^2\left[f(0)g'(0)-f'(0)g(0)\right]\neq 0$$

(Vii) $p(x)=x^2,\ -1<x<1,\ u(-1)=u(1),\ u'(-1),\ u'(1)$

$$1\left[f'(1)g(1)-g'(1)f(1)\right]+1\left[f'(-1)g(-1)-f(-1)g'(-1)\right]$$

$$f'(1)g(1)-g'(1)f(1)+f'(1)g(1)-f(1)g'(1)=2f'(1)g(1)-2f(1)g'(1)\neq 0$$

1/ ج/ تسمى المسألة الحدية إذا تحقق أحد الشروط التالية مسألة ستورم ليوفيل الشاذة

$a\to-\infty$ أو $b\to\infty \Rightarrow [a,b]$ غير محدودة

$p(a)\leq 0,\ P(b)\leq 0$

أو

$p(x)\leq 0$ for $\forall x(a,b)$

(29)

$(iii)\ p(x) = \sin x \quad 0 < x < \pi/2 \ , \ u(0) = 1, \ u(\pi/2) = 0$

$p(0) = \sin 0 = 0$ ح.ت

كمل مانه الشروط الدورية لكن ...

$(v)\ p(x) = x^2, \ 0 < x < b, \ u'(0) = u(b); \ u'(b) = u(b)$

ح.ب

$p(0) = 0$

$(vi)\ p(x)\ x^2, \ 0 < x < b$

$u'(0) = u(b), \ u'(b) = u(0)$

ح.ب

$p(0) = 0$

$(vii)\ p(x) = x^2, \ -1 < x < 1 \ , \ u(-1) = u(1), \ u'(-1) = u'(1)$

ح.ب

$0 \in (-1, 1) \ : \ p(0) = 0$

الدالة مربعة بالتالي لا تحقق ساكة (لا)

3 أوجد القيم والدوال الذاتية للمسألة

$[(x+3)^2 y']' + \lambda y = 0, \ -2 \leq x \leq 1$

$y(-2) = y(1) = 0$

الحل:

$a_2 x^2 y'' + a_1 x y' + a_0 y = 0 \quad , \ y = x^m$

$\Rightarrow y' = m x^{m-1} \ , \ y'' = m(m-1) x^{m-2}$

$[a_2 m(m-1) + a_1 m + a_0] x^m = 0 \quad , \ x^m \neq 0$

$\Rightarrow [a_2 m(m-1) + a_1 m + a_0] = 0 \Rightarrow m_1, m_2$

$(i)\ m_1 \neq m_2 \Rightarrow y_c = c_1 x^{m_1} + c_2 x^{m_2}$

$(ii)\ m_1 = m_2 \Rightarrow y_c = c_1 x^{m_1} + c_2 x^{m_1} \ln x$

$(iii)\ m_{1,2} = \alpha \pm i\beta \Rightarrow y_c = x^{\alpha}[c_1 \cos \beta \ln x + c_2 \sin \beta \ln x]$

٢-

بسم الله

$$(x+3)^2 y'' + 2(x+3)y' + \lambda y = 0 \quad (*)$$

$$x+3 = t \Rightarrow \frac{d}{dx} = \frac{d}{dt}$$

$$(*) \Rightarrow t^2 \ddot{y} + 2t\dot{y} + \lambda y = 0$$

$$y = t^m \Rightarrow \dot{y} = m t^{m-1}, \quad \ddot{y} = m(m-1)t^{m-2}$$

$$m(m-1) + 2m + \lambda = 0$$

$$\Rightarrow m^2 + m + \lambda = 0 \Rightarrow m_{1,2} = \frac{-1 \pm \sqrt{1-4\lambda}}{2}, \; ; \; 1-4\lambda < 0$$

لأنه لو $1-4\lambda \geq 0$ نحصل على حلول تافهة

$$y_c = t^{-1/2}\left[c_1 \cos\left(\frac{\sqrt{1-4\lambda}}{2}\ln t\right) + c_2 \sin\left(\frac{\sqrt{1-4\lambda}}{2}\ln t\right)\right]$$

$$\Rightarrow y_c = (x+3)^{-1/2}\left[c_1 \cos\sqrt{\tfrac{1}{4}-\lambda}\,\ln(x+3) + c_2 \sin\sqrt{\tfrac{1}{4}-\lambda}\,\ln(x+3)\right]$$

نطبق الشروط الحدية

$$y(-2) = 0 = 1\left[c_1 \cos(0) + c_2 \sin(0)\right]$$

$$\Rightarrow \boxed{0 = c_1}$$

$$y(1) = 0, \Rightarrow (4)^{-1/2}\left[c_2 \sin\sqrt{\tfrac{1}{4}-\lambda}\,\ln 4\right] = 0$$

$c_2 \neq 0$ لأنه حل تافه

$$\Rightarrow \sin\left(\sqrt{\tfrac{1}{4}-\lambda}\,\ln 4\right) = 0, \Rightarrow \sqrt{\tfrac{1}{4}-\lambda}\,\ln 4 = n\pi \; ; \; n \in \mathbb{N}$$

$$\sqrt{\tfrac{1}{4}-\lambda} = \frac{n\pi}{\ln(4)} \Rightarrow \tfrac{1}{4} - \lambda = \frac{n^2\pi^2}{(\ln(4))^2} \Rightarrow \lambda_n = \frac{1}{4} - \left(\frac{n\pi}{\ln(4)}\right)^2 ; \; n \in \mathbb{N}$$

هي القيم الذاتية

أما الدوال الذاتية فهي:

$$y_n(x) = (x+3)^{-1/2} \sin\left[\frac{n\pi}{\ln 4}\ln(x+3)\right] ; \; n \in \mathbb{N}$$

5

$$(pu')' + \cancel{}\, r\,u + \lambda u = 0, \quad a < x < b \quad (*)$$

$$u(a) = u(b) = 0$$

Scanned by CamScanner

۲۰۱۶ ، ج ، تمارين ، تمارين

٣١

$$\lambda \int_a^b |u|^2 dx = \int_a^b p|u'|^2 dx - \int_a^b r|u|^2 dx \qquad (i)$$

$$\lambda u = -[(pu')' + ru]$$

$$\lambda u^2 = -[(pu')' + ru]u \qquad (*)$$

$$\lambda \int_a^b u^2 dx = -\int_a^b [(pu')' + ru] u\, dx$$

$$= \underbrace{-\int_a^b (pu')' u\, dx}_{I_1 \text{ بالتجزئة}} - \int_a^b r u^2 dx$$

$$= \int_a^b (pu')' u\, dx$$

بوضع $t = u \Rightarrow dt = u' dx$

$dv = (pu')' dx \Rightarrow v = pu'$

$$= [upu']_a^b - \int_a^b p(u')^2 dx = u(b)p(b)u'(b) - u(a)p(a)u'(a) - \int_a^b p(u')^2 dx$$

$$\Rightarrow \lambda \int_a^b (u)^2 dx = \int_a^b p(u')^2 - \int_a^b r(u)^2 dx$$

$$\Rightarrow \lambda \int_a^b |u|^2 dx = \int_a^b p|u'|^2 dx - \int_a^b r|u|^2 dx$$

(ii) $p(x) \geq 0$ ، $r(x) \leq c$ ، أثبت أن

$$\lambda \geq -c$$

واجب

من فرض (i) نجد أن

$$\lambda \int_a^b |u|^2 dx = \int_a^b p|u'|^2 dx - \int_a^b r|u|^2 dx \geq \int_a^b p|u'|^2 - \int_a^b c|u|^2 dx$$

لأن

$$-\int ru^2 \geq -\int cu^2 \Leftarrow \int ru^2 \leq \int cu^2 \Leftarrow r(x)u^2 \leq cu^2 \Leftarrow r(x) \leq c$$

$$\Rightarrow \lambda \int_a^b |u|^2 dx = \int_a^b p|u'|^2 dx - c\int_a^b c|u|^2 dx$$

حيث أن $|u|^2 \geq 0$ بـ c $\int_a^b |u|^2 dx > 0$ فلذا بالقسمة عليه

نجد أن

بالعكس.

$$\lambda = \frac{\int_a^b p|u'|^2 dx}{\int_a^b |u|^2 dx} - c$$

وحيث أن $p \geq 0 , |u'|^2 \geq 0$ إذن $\int p|u'|^2 \geq 0$

$$\Rightarrow \frac{\int_a^b p|u'|^2}{\int_a^b |u|^2} \geq 0 \Rightarrow \lambda \geq -c$$