[3] Find ~~Show that~~ $\sum_{k=0}^{100} e^{kz}$:

$*$ If $z=0$

$\Rightarrow e^{k(0)} = e^{0} = 1$

$\because \sum_{k=0}^{100} (1) = \underbrace{1+1+\cdots+1}_{101 \text{ times}} = 101$

$*$ at $z \neq 0$ " $e^{z} \neq 1$ "

$\sum_{k=0}^{100} e^{kz} = e^{0} + (e^{z})^{1} + (e^{z})^{2} + \cdots + (e^{z})^{100}$ " as $1+z+\cdots+z^{n} = \frac{z^{n+1}-1}{z-1}$ "

$= \frac{(e^{z})^{101}-1}{e^{z}-1}$

Hence, $\sum_{k=0}^{100} e^{zk} = \begin{cases} \frac{e^{101z}-1}{e^{z}-1} & \text{if } z \neq 0 \\ 101 & \text{if } z = 0 \end{cases}$

[4] $w = re^{i\theta}$ show that $e^{\ln r + i\theta} = w$

L.H.S $e^{\ln r + i\theta} = e^{\ln r} \cdot e^{i\theta}$

$= re^{i\theta}$

$= w$ R.H.S

[5] (a) $\exp(2 + i\pi/4) = e^{2+\frac{\pi}{4}i} = e^{2} \cdot e^{\frac{\pi}{4}i} = e^{2}\left[\frac{1}{\sqrt{2}} + i/\sqrt{2}\right] = \frac{e^{2}}{\sqrt{2}} + \frac{e^{2} i}{\sqrt{2}}$

(b) $\frac{\exp(1+3\pi i)}{\exp(-1+i\pi/2)}$

$= e^{2+\frac{5\pi}{2}i} = e^{2}[i]$

$= 0 + ie^{2}$

(c) $\sin(2i) = \frac{e^{2i(i)} - e^{-2i(i)}}{2i} = \frac{e^{-2} - e^{2}}{2i} = \left(\frac{e^{2} - e^{-2}}{2}\right)i$

(f) $\cosh(i\frac{\pi}{2}) = \frac{e^{i\frac{\pi}{2}} + e^{-i\frac{\pi}{2}}}{2} = \frac{i - i}{2} = 0$

3.1

1) $e^{z} = \frac{1+i}{\sqrt{2}} \longleftrightarrow z = (\pi/4 + 2k\pi)\, i \quad , k \in \mathbb{Z}$

Let $z = x+iy$

$(\Rightarrow)$ suppose $e^{z} = \frac{1+i}{\sqrt{2}}$

$\Rightarrow |e^{z}| = \left|\frac{i+1}{\sqrt{2}}\right|$

$\Rightarrow e^{x} = 1$

$\Rightarrow x = 0$

so, $\boxed{z = iy}$

$e^{z} = e^{iy} = \cos y + i \sin y$

since $e^{iy} = \frac{1+i}{\sqrt{2}} = 1/\sqrt{2} + i/\sqrt{2}$

$\Rightarrow \cos y = 1/\sqrt{2}$

$\sin y = 1/\sqrt{2} \Rightarrow y = \pi/4 + 2k\pi, \ k \in \mathbb{Z}$

$\Rightarrow z = i(\pi/4 + 2k\pi)$

$(\Leftarrow)$ suppose that $z = (\pi/4 + 2k\pi)\, i$

$\Rightarrow e^{z} = e^{(\pi/4 + \pi k)i}$

$= \cos(\pi/4 + 2k\pi) + i \sin(\pi/4 + 2k\pi)$

$= 1/\sqrt{2} + i/\sqrt{2}$

$= (1+i)/\sqrt{2}$

2) If $f(z) = e^{z} - (1 + z + z^2/2 + z^3/6)$ Show that $f^{(3)}(0) = 0$

$f'(z) = e^{z} - (1 + z + z^2/2)$

$f''(z) = e^{z} - (1 + z)$

$f^{(3)}(z) = e^{z} - 1$

$\therefore f^{(3)}(0) = e^{0} - 1 = 1 - 1 = 0$

[7] Show $e^{iz} = \cos z + i \sin z$

$$\text{R.H.S} = \cos z + i \sin z$$
$$= \left(\frac{e^{iz} + e^{-iz}}{2}\right) + \cancel{i}\left(\frac{e^{iz} - e^{-iz}}{2\cancel{i}}\right)$$
$$= \frac{1}{2}\left(e^{iz} + \cancel{e^{-iz}} + e^{iz} - \cancel{e^{-iz}}\right)$$
$$= \frac{1}{2}(\cancel{2}e^{iz})$$
$$= e^{iz}$$
$$= \text{L.H.S}$$

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[9] (a) $w = e^{\pi z^2}$

$$\frac{dw}{dz} = 2\pi z e^{\pi z^2}$$

(b) $w = \cos(2z) + i \sin(1/z)$

$$\frac{dw}{dz} = -2\sin(2z) - \frac{i}{z^2}\cos\left(\frac{1}{z}\right)$$

---

[10] why $f(z) = \sin(z^2) + e^{-z} + iz$ entire ?!

since z^2 entire "proved"

and $\sin z$ entire

$\Rightarrow \sin(z^2)$ entire

also, e^{-z} entire and iz entire

$\Rightarrow f(z)$ entire "By Th."

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[11] why $\mathrm{Re}\left(\frac{\cos z}{e^z}\right)$ is harmonis in $\mathbb{C}$?

Since $\cos z$ is entire and $e^z \neq 0$ $\forall z$ entire

$\Rightarrow \cos z / e^z$ entire "analytic on $\mathbb{C}$"

so, Real Part of analytic function is a harmonic

12 (a) $\cosh^2(z) - \sinh^2(z) = 1$

R.H.S $\cosh^2(z) - \sinh^2(z)$

$$= \left(\frac{e^{z}+e^{-z}}{2}\right)^2 - \left(\frac{e^{z}-e^{-z}}{2}\right)^2$$

$$= \frac{1}{4}\left[e^{2z} + 2e^{-z}e^{z} + e^{-2z}\right] - \left[e^{2z} - 2e^{z}e^{-z} + e^{-2z}\right]$$

$$= \frac{1}{4}\left[\cancel{e^{2z}} + 2 + \cancel{e^{-2z}} - \cancel{e^{2z}} + 2 - \cancel{e^{-2z}}\right]$$

$$= \frac{1}{4}[4]$$

$$= 1$$

R.H.S

(b) $\sinh(z_1+z_2) = \sinh z_1 \cosh z_2 + \cosh z_1 \sinh z_2$

R.H.S $\sinh z_1 \cosh z_2 + \cosh z_1 \sinh z_2$

$$= \left(\frac{e^{z_1}-e^{-z_1}}{2}\right)\left(\frac{e^{z_2}+e^{-z_2}}{2}\right) + \left(\frac{e^{z_1}+e^{-z_1}}{2}\right)\left(\frac{e^{z_2}-e^{-z_2}}{2}\right)$$

$$= \frac{1}{4}\left[e^{z_1+z_2} + \cancel{e^{z_1-z_2}} - \cancel{e^{-z_1+z_2}} - e^{-z_1-z_2}\right] + \frac{1}{4}\left[e^{z_1+z_2} - \cancel{e^{z_1-z_2}} + \cancel{e^{-z_1+z_2}} - e^{-z_1-z_2}\right]$$

$$= \frac{1}{\cancel{4}_2}\left[\cancel{2}e^{z_1+z_2} - 2e^{-(z_1+z_2)}\right]$$

$$= \frac{e^{z_1+z_2} - e^{-(z_1+z_2)}}{2}$$

$$= \sinh(z_1+z_2) = L.H.S$$

(c) $\cosh(z_1+z_2) = \cosh z_1 \cosh z_2 + \sinh(z_1)\sinh(z_2)$

R.H.S $\cosh(z_1)\cosh(z_2) + \sinh(z_1)\sinh(z_2)$

$$= \left(\frac{e^{z_1}+e^{-z_1}}{2}\right)\left(\frac{e^{z_2}+e^{-z_2}}{2}\right) + \left(\frac{e^{z_1}-e^{-z_1}}{2}\right)\left(\frac{e^{z_2}-e^{-z_2}}{2}\right)$$

$$= \frac{1}{4}\left[e^{z_1+z_2} + \cancel{e^{z_1-z_2}} + \cancel{e^{-z_1+z_2}} + e^{-z_1-z_2}\right] + \frac{1}{4}\left[e^{z_1+z_2} - \cancel{e^{z_1-z_2}} - e^{-z_1+z_2} + e^{-(z_1+z_2)}\right]$$

$$= \frac{1}{4}\left[2e^{z_1+z_2} + 2e^{-(z_1+z_2)}\right]$$

$$= \frac{e^{z_1+z_2} + e^{-(z_1+z_2)}}{2} = \cosh(z_1+z_2) = L.H.S$$

13 (a) $\sin(x+iy) = \sin x \cosh y + i \cos x \sinh y$

R.H.S $\sin x \cosh y + i \cos x \sinh y$.

$$= \left(\frac{e^{xi}-e^{-xi}}{2i}\right)\left(\frac{e^{y}+e^{-y}}{2}\right)+i\left(\frac{e^{xi}+e^{-xi}}{2}\right)\left(\frac{e^{y}-e^{-y}}{2}\right)$$

$$= \left(\frac{e^{xi+y}+e^{xi-y}-e^{-xi+y}-e^{-xi-y}}{4i}\right)+\frac{i}{4}\left[\frac{e^{xi+y}-e^{xi-y}+e^{-xi+y}-e^{-xi-y}}{}\right]$$

$$= \left[\frac{e^{xi+y}+\cancel{e^{xi-y}}-e^{-xi+y}-\cancel{e^{-xi-y}}}{4i}\right]+\frac{1}{4i}\left[-\cancel{e^{xi+y}}+e^{xi-y}-e^{-xi+y}+\cancel{e^{-xi-y}}\right]$$

$$= \frac{1}{4i}\left[\cancel{2}e^{xi+y}-\cancel{2}e^{-xi+y}\right]$$

$$= \frac{e^{(x+iy)i}-e^{-(x+yi)i}}{2i}$$

$= \sin(x+yi) = L.H.S$

(b) ~~similarly~~ $\cos(x+iy) = \cos(x)\cos(iy) - \sin x \sin(iy)$
$= \cos x \cosh y + i \sin x \sinh y$

$\boxed{\text{as: } \cos iy = \cosh y,\ -i\sin z = \sinh z}$

(iii) ⓐ e^{iz} is periodic with period $2\pi \iff f(z+2\pi) = f(z)$

$\overset{??}{\iff} e^{i(z+2\pi)} = e^{iz}$

$f(z+2\pi) = e^{i(z+2\pi)} = e^{zi+2\pi i} = e^{zi}\cdot e^{2\pi i} = e^{zi} = f(z)$

$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad 1$

ⓑ $f(z+\pi) \overset{??}{=} f(z)$

$f(z+\pi) = \tan(z+\pi)$

$$= \frac{\sin(z+\pi)}{\cos(z+\pi)} = \frac{\sin z \cos\pi + \cos z \sin\pi}{\cos z\cos\pi - \sin z \sin\pi} = \frac{-\sin z}{-\cos z} = \tan z$$

$= f(z)$

ⓒ $f(z+2\pi i) \overset{??}{=} f(z)$

$f(z+2\pi i) = \sinh(z+2\pi i)$

$$= \frac{e^{z+2\pi i}-e^{-z-2\pi i}}{2}$$

$$= \frac{e^{z}\cdot e^{2\pi i}-e^{-z}\cdot e^{-2\pi i}}{2}$$

$$= \frac{e^{z}-e^{-z}}{2} = \sinh z = f(z)$$

15] $\cos z = 0 \longleftrightarrow z = \pi/2 + \cancel{k}k\pi$

let $z = x + iy$

$(\Rightarrow)$ suppose that $\cos z = 0$

$$\Rightarrow \frac{e^{iz} + e^{-iz}}{2} = 0$$

$$\Rightarrow e^{iz} + e^{-iz} = 0$$

$$\Rightarrow e^{iz} = -e^{-iz}$$

$$\Rightarrow |e^{iz}| = |-e^{-iz}|$$

$$\Rightarrow -y = y$$

$$\Rightarrow y = 0$$

so, $\boxed{z = x}$

$$\therefore \cos z = \cos(x) = \frac{e^{xi} + e^{-xi}}{2} = 0$$

$$\Rightarrow e^{ix} = -e^{-ix}$$

$$\Rightarrow e^{2ix} = -1$$

$$\Rightarrow \cos(2x) + i\sin(2x) = -1$$

$$\Rightarrow \cos 2x = -1 \ \wedge \ \sin(2x) = 0$$

$$\Rightarrow 2x = \pi + 2k\pi \Rightarrow x = \pi/2 + \pi k, \ k \in \mathbb{Z}$$

$\boxed{\therefore z = \pi/2 + \pi k}$

$(\Leftarrow)$ suppose $z = \pi/2 + k\pi$

$$\therefore \cos z = \cos(\pi/2 + k\pi) = 0$$

17 (a) $e^{4z}=1$

let $z=x+iy$

$\because e^{4z}=1$

$\Rightarrow |e^{4z}|=1$

$\Rightarrow e^{4x}=1$

$\Rightarrow 4x=0$

$\therefore \boxed{x=0}$

$\boxed{\therefore z=yi}$

$\because e^{4yi}=1$

$\leftrightarrow \cos 4y+i\sin 4y=1$

$\leftrightarrow \cos 4y=1$, $\sin 4y=0 \Rightarrow 4y=2k\pi$, $k\in\mathbb{Z}$

$\Rightarrow y=\frac{\pi}{2}k$, $k\in\mathbb{Z}$

$\boxed{\therefore z=\frac{\pi}{2}ki}$

(b) $e^{iz}=3$

since $3=e^{\ln(3)}$

$\therefore e^{iz}=e^{\ln(3)}$

$\leftrightarrow iz=\ln(3)+2k\pi i$

$\Rightarrow z=\frac{\ln(3)}{i}+2k\pi$

$=-i\ln(3)+2k\pi$

$=i\ln(1/3)+2k\pi$

(c) $\cos z=i\sin z$

$\Rightarrow \frac{e^{iz}+e^{-iz}}{\cancel{2}}=\cancel{i}\frac{e^{iz}-e^{-iz}}{2\cancel{i}}$

$\Rightarrow \cancel{e^{iz}}+e^{-iz}=\cancel{e^{iz}}-e^{-iz}$

$\Rightarrow 2e^{-iz}=0$ X

So, there is no z such that $\cos z=i\sin z$

18 let $f(z)=\sin(z)$

since $\sin z$ entire $\Rightarrow$ $f'=\cos z$, $\sin z$ analytic on $z=0$, $f'(0)=0$

(i.e) $\lim_{z\to 0}\frac{f(z)-f(0)}{z-0}=0$

$\Rightarrow \lim_{z\to 0}\frac{\sin(z)-\sin(0)}{z-0}$

$\boxed{=\lim_{z\to 0}\frac{\sin z}{z}=0}$

(b) $f(z)=\cos z \to f'(z)=-\sin z$

$f'(0)=0$

$\therefore \lim_{z\to 0}\frac{\cos(z)-\cos(0)}{z-0}=0$

$\boxed{\Rightarrow \lim_{z\to 0}\frac{\cos(z)-1}{z}=0}$

$\underline{\boxed{19}}$ e^z is 1-1 on any disk of radius $\underline{\pi}$ "(i.e) on $\underset{\sim}{|z_1 - z_2| < \pi}$"

$\underline{\text{Ans}}$: let $e^{z_1} = e^{z_2}$ and we will show $z_1 = z_2$

Now, $e^{z_1} = e^{z_2}$

$\Rightarrow z_1 = z_2 + 2k\pi i \;,\; k \in \mathbb{Z}$

$\Rightarrow z_1 - z_2 = 2k\pi i$

$\Rightarrow |z_1 - z_2| = 2k\pi$ $\leftarrow$ +ve or $\underline{0}$ "as $|z_1 - z_2| < \pi$"

$\Rightarrow |z_1 - z_2| = 0$

$\Rightarrow z_1 = z_2$

$\boxed{21}$ (a) $\sin z$ entire $\Rightarrow \sin^2 z$ entire

$\cos z$ " " , $\cos^2 z$ "

$\Rightarrow \sin^2 z + \cos^2 z$ entire

$\underline{(b)}$ $f'(z) = 2\sin z \cancel{\cos z} - \cancel{2\cos z \sin z}$

$= 0 \quad \forall z$

(c) since $f'(z) = 0 \quad \forall z \in \mathbb{C}$ $\leftarrow$ domain

$\Rightarrow f$ is constant

(d) $f(0) = \sin^2(0) + \cos^2(0) = 1$

(e) since f constant and $f(0) = 1$

$\Rightarrow f(z) = 1 \quad \forall z \in \mathbb{C}$

$\therefore \boxed{\sin^2 z + \cos^2 z = 1}$