Measure Theory

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Introduction on Measures

Definition

Let $(X, \mathscr{A})$ be a measurable space. A measure (or a positive measure) on X is a function $\mu\colon \mathscr{A} \to [0, \infty]$ such that

1. $\mu(\emptyset) = 0$;
2. For any disjoint sequence $(A_n)_n \in \mathscr{A}$, (Countable additivity)

$$\mu(\bigcup_{n=1}^{+\infty} A_n) = \sum_{n=1}^{+\infty} \mu(A_n). \tag{1}$$

The set $(X, \mathscr{A}, \mu)$ will be called a measure space.

Examples

1. Let X be any non empty set and let $\mathscr{A} = \mathscr{P}(X)$. For $A \in \mathscr{A}$, we define $\mu(A) = \#A$ the number of elements of A if A is finite and equal to $+\infty$ otherwise. ($\#A$ is called also the cardinal of A). μ is then a measure on $\mathscr{A}$. This measure is called the **counting measure**.
2. $\delta_a(A) = 1$ if $a \in A$ and 0 otherwise. The measure δ_x is called the **point mass** at a or **the Dirac measure** at a.

3. Let μ defined on $\mathscr{P}(\mathbb{R})$ by

$$\mu(A) = \begin{cases} 0 & \text{if } A \text{ is finite} \\ +\infty & \text{otherwise} \end{cases}$$

μ is finite additive but not countably additive since $\mathbb{N} = \bigcup_{n=1}^{+\infty}\{n\}$, but $\mu(\mathbb{N}) = +\infty \neq \sum_{n=1}^{+\infty} \mu(\{n\}) = 0$. μ is not a measure.

Theorem

Let $(X, \mathscr{A}, \mu)$ be a measure space. The measure μ fulfills the following basic properties

1. μ is finitely additive For any finite subsets $A_1, \ldots, A_n \in \mathscr{A}$ of disjoints elements of $\mathscr{A}$, $\mu(\cup_{j=1}^n A_j) = \sum_{j=1}^n \mu(A_j)$.
2. μ is monotone If $A, B \in \mathscr{A}$ with $A \subset B$, then $\mu(A) \leq \mu(B)$.
3. μ is countably subadditive If $(A_n)_n \in \mathscr{A}$ and $A = \cup_{n=1}^{+\infty} A_n$, then

$$\mu(A) \leq \sum_{n=1}^{+\infty} \mu(A_n).$$

Definition

4. (Continuity from below:) If $(A_n)_n$ is an increasing sequence in $\mathscr{A}$, and $A = \cup_{n=1}^{+\infty} A_n$, then $\mu(A) = \lim_{n \to +\infty} \mu(A_n)$.
5. μ is subtractive If $A, B \in \mathscr{A}$ and $A \subset B$ and $\mu(B) < +\infty$, then $\mu(B \setminus A) = \mu(B) - \mu(A)$. ($\mu(A) < \infty$ suffices).
6. (Continuity from above:) If $(A_n)_n$ is a decreasing sequence in $\mathscr{A}$ with $\mu(A_1) < \infty$, then $\mu(A) = \lim_{n \to +\infty} \mu(A_n)$, with $A = \cap_{n=1}^{+\infty} A_n = \lim_{n \to +\infty} A_n$.

Proof

1. This property is obvious.
2. $B = A \cup (B \setminus A)$, then $\mu(B) = \mu(A) + \mu(B \setminus A) \geq \mu(A)$. We use the property 2) of the definition of measure.
3. Let $B_1 = A_1$, and $B_n = A_n \setminus \bigcup_{j=1}^{n-1} B_j$, for $n \geq 2$. The sets $(B_n)_n$ are disjoints and $A = \bigcup_{n=1}^{+\infty} B_n = \bigcup_{n=1}^{+\infty} A_n$. So
$$\mu(A) = \sum_{n=1}^{+\infty} \mu(B_n) \leq \sum_{n=1}^{+\infty} \mu(A_n).$$

4. Let $(B_n)_n$ as in 3). Since $\bigcup_{j=1}^{n} A_j = \bigcup_{j=1}^{n} B_j$, then

$$\mu(A) = \mu(\bigcup_{n=1}^{+\infty} A_n) = \mu(\bigcup_{n=1}^{+\infty} B_n) = \sum_{n=1}^{+\infty} \mu(B_n) = \lim_{n\to\infty} \sum_{j=1}^{n} \mu(B_j)$$
$$= \lim_{n\to\infty} \mu(\bigcup_{j=1}^{n} B_j) = \lim_{n\to\infty} \mu(\bigcup_{j=1}^{n} A_j) = \lim_{n\to\infty} \mu(A_n$$

5. $\mu(B \setminus A) + \mu(A) = \mu(B)$. If $\mu(A) < \infty$ then $\mu(B \setminus A) = \mu(B) - \mu(A)$.
6. Apply 3) to the sequence $(A_1 \setminus A_n)_n$.

□

Exercise

Let $(X, \mathscr{A})$ be a measurable space and $\mu\colon \mathscr{A} \longrightarrow [0, +\infty]$ a set function. Prove that μ is a measure if and only if

i) $\mu(\emptyset) = 0$

ii) $\mu(A \cup B) = \mu(A) + \mu(B)$, if $A \cap B = \emptyset$.

iii) If $(A_n)_n$ is an increasing sequence in the σ-algebra $\mathscr{A}$, then

$$\mu(\bigcup_{n=1}^{+\infty} A_n) = \lim_{n \to +\infty} \mu(A_n).$$

Solution

If μ is a measure, the properties i) and ii) are evident. Let now $(A_n)_n$ be an increasing sequence of the σ-algebra $\mathscr{A}$, then the sequence $(B_n)_n$ defined by $B_1 = A_1$ and $B_n = A_n \setminus \cup_{j=1}^{n-1} A_j$ is disjoint and $\bigcup_{n=1}^{+\infty} A_n = \bigcup_{n=1}^{+\infty} B_n$. Then

$$\begin{aligned}\mu\left(\bigcup_{n=1}^{+\infty} A_n\right) &= \sum_{n=1}^{+\infty} \mu(B_n) = \lim_{n\to+\infty} \sum_{j=1}^{n} \mu(B_j) \\ &= \lim_{n\to+\infty} \mu(\cup_{j=1}^{n} B_j) = \lim_{n\to+\infty} \mu(A_n).\end{aligned}$$

Conversely, if μ fulfills the properties i), ii) and iii) and $(A_n)_n$ a sequence of disjoint measurable sets. Let $B_n = \bigcup_{j=1}^{n} A_j$, for $n \in \mathbb{N}$.

The sequence $(B_n)_n$ is increasing and $\bigcup_{n=1}^{+\infty} A_n = \bigcup_{n=1}^{+\infty} B_n$. Then

$$\mu(\bigcup_{n=1}^{+\infty} A_n) = \lim_{n\to+\infty} \mu(B_n) = \lim_{n\to+\infty} \sum_{j=1}^{n} \mu(A_j) = \sum_{n=1}^{+\infty} \mu(A_n).$$

Definition

1. We say that the measure μ is **finite** if $\mu(X) < +\infty$.
2. We say that the measure μ is σ-**finite** if there exists an increasing sequence $(A_n)_n$ of measurable subsets of finite measure and $\displaystyle\bigcup_{n=1}^{+\infty} A_n = X$.
3. A probability measure is a measure on $(X, \mathscr{A})$ such that $\mu(X) = 1$. In this case the $\sigma-$algebra $\mathscr{A}$ is called the space of events.

Remark

Let $(X, \mathscr{A})$ be a measurable space. We denote by $\mathscr{M}(X, \mathscr{A})$ or $\mathscr{M}(X)$ the set of measures on the measurable space $(X, \mathscr{A})$. We have the following properties
The set $\mathscr{M}(X)$ is a convex cone: If μ_1 and μ_2 are in $\mathscr{M}(X)$ and $\lambda \in \mathbb{R}^+$, then $\mu_1 + \mu_2$, $\lambda\mu_1$ are measures. We order the set $\mathscr{M}(X)$ by the relationship

$$\mu_1 \leq \mu_2 \iff \mu_1(A) \leq \mu_2(A);\ \forall A \in \mathscr{A}.$$

Theorem

Let $(X, \mathscr{A})$ be a measurable space. If $(\mu_n)_n$ is an increasing sequence of measures, then the set function $\mu\colon \mathscr{A} \longrightarrow [0, +\infty]$ defined by $\mu(A) = \lim_{n\to+\infty} \mu_n(A) = \sup_n \mu_n(A)$ for any $A \in \mathscr{A}$ is a measure on X.

Proof

It is clear that $\mu(\emptyset) = 0 = \lim_{n\to+\infty} \mu_n(\emptyset)$, and if A, B are two disjoints measurable sets, we have

$$\mu(A \cup B) = \lim_{n\to+\infty} \mu_n(A) + \lim_{n\to+\infty} \mu_n(B) = \mu(A) + \mu(B).$$

Let now $(A_n)_n$ be an increasing sequence of $\mathscr{A}$ and $A = \bigcup_{n=1}^{+\infty} A_n$. We have

$\mu_j(A_n) \leq \mu(A_n) \leq \mu(A)$. Then
$\mu_j(A) = \lim_{n\to+\infty} \mu_j(A_n) \leq \lim_{n\to+\infty} \mu(A_n) \leq \mu(A)$
and

$$\mu(A) = \lim_{j\to+\infty} \mu_j(A) \leq \lim_{n\to+\infty} \mu(A_n) \leq \mu(A).$$

Then $\mu(A) = \lim_{n\to+\infty} \mu(A_n)$.

Lebesgue-Stieltjes Measure

Definition

A Lebesgue-Stieltjes measure on $\mathbb{R}$ is a measure on the Borel $\sigma-$algebra $\mathscr{B}_{\mathbb{R}}$ such that $\mu(I) < +\infty$ for all bounded interval I.

Proposition

Let μ be a Lebesgue-Stieltjes measure on $\mathbb{R}$. Define $f : \mathbb{R} \longrightarrow \mathbb{R}$ by $f(b) - f(a) = \mu]a, b]$. For example, fix $f(0)$ arbitrary and set

$$\begin{cases} f(x) - f(0) = \mu]0, x], & \text{if } x > 0, \\ f(0) - f(x) = \mu]x, 0], & \text{if } x < 0. \end{cases}$$

The function f is right continuous and increasing.

Proof

Let $a < b$, $f(b) - f(a) = \mu]a, b] \geq 0$. Also, if $(x_n)_n$ is a decreasing sequence and converges to x, then $\lim_{n\to+\infty} \mu]x, x_n] = \lim_{n\to+\infty} f(x_n) - f(x) = 0$.

□

Remarks

1. Let $a \in \mathbb{R}$,

$$\mu\{a\} = \lim_{n\to+\infty} \mu]a-\frac{1}{n}, a] = \lim_{n\to+\infty} f(a)-f(a-\frac{1}{n}) = f(a)-f(a-).$$

   Then f is continuous at a if and only if $\mu\{a\} = 0$.

2. $\mu([a, b]) = f(b) - f(a-)$,
   $\mu(]a, b[) = f(b-) - f(a)$,
   $\mu([a, b[) = f(b-) - f(a-)$,
   $\mu(]a, b]) = f(b) - f(a)$.

Theorem

Let $f : \mathbb{R} \to \mathbb{R}^+$ an increasing right-continuous function. There is a unique measure μ on $\mathscr{B}_{\mathbb{R}}$ such that $\mu]a, b] = f(b) - f(a)$.

Proof

We know that $\sigma(\mathcal{I}) = \mathscr{B}_{\mathbb{R}}$.
Let $\mathcal{I} = \{]a, b] : -\infty < a < b < \infty\}$. Set $f(+\infty) = \lim_{x\to+\infty} f(x)$ and $f(-\infty) = \lim_{x\to-\infty} f(x)$. These quantities exist since f is increasing. Define for any $\mu(]a, b]) = f(b) - f(a)$, for any $-\infty \leq a < b \leq \infty$. Suppose $]a, b] = \cup_{j=1}^{n}]a_j, b_j]$, where the union is disjoint, then $\mu(]a_j, b_j]) = f(b_j) - f(a_j)$ and

$$\sum_{j=1}^{n} \mu(]a_j, b_j]) = \sum_{j=1}^{n} f(b_j) - f(a_j) = f(b) - f(a) = \mu(]a, b]),$$

which proves that condition (i) holds.

For (ii), let $a, b \in \mathbb{R}$ and $]a, b] \subset \bigcup_{j=1}^{+\infty}]a_j, b_j]$ where the union is disjoint. (We can also order them if we want.) By right continuity of f, given $\varepsilon > 0$ there is $\delta > 0$ such that $f(a+\delta) - f(a) < \varepsilon$, or $f(a+\delta) < f(a) + \varepsilon$.
Similarly, there is $\eta_j > 0$ such that $f(b_j + \eta_j) < f(b_j) + \frac{\varepsilon}{2^j}$, for all j. Now, $\{]a_j, b_j + \eta_j[\}$ forms an open cover for $[a+\delta, b]$. By compactness, there is a finite sub-cover. Thus,
$[a+\delta, b] \subset \cup_{j=1}^{N}]a_j, b_j + \eta_j[$ and $]a+\delta, b] \subset \cup_{j=1}^{N}]a_j, b_j + \eta_j]$.
Therefore

$$\begin{aligned} f(b)-f(a+\delta) &\leq \sum_{j=1}^{N}\mu(]a_j,b_j+\eta_j]) = \sum_{j=1}^{N} f(b_j+\eta_j)-f(a_j) \\ &= \sum_{j=1}^{N} f(b_j+\eta_j)-f(b_j)+f(b_j)-f(a_j) \\ &\leq \sum_{j=1}^{N}\frac{\varepsilon}{2^j}+\sum_{j=1}^{+\infty}(f(b_j)-f(a_j)) \\ &\leq \varepsilon+\sum_{j=1}^{+\infty}(f(b_j)-f(a_j)). \end{aligned}$$

Therefore,

$$\begin{aligned}\mu(]a,b]) &= f(b) - f(a)\\ &\leq 2\varepsilon + \sum_{j=1}^{+\infty}(f(b_j) - f(a_j))\\ &\leq 2\varepsilon + \sum_{j=1}^{+\infty}\mu(]a_j,b_j]).\end{aligned}$$

If $]a,b] \subset \bigcup_{j=1}^{+\infty}]a_j,b_j]$, a and b arbitrary, and $]c,d] \subset]a,b]$ for any $c,d \in \mathbb{R}$, we have by above

$$f(d) - f(c) \leq \sum_{j=1}^{+\infty} (f(b_j) - f(a_j))$$

and the result follows by taking limits. □

Complete Measure Space

Definition

Let $(X, \mathscr{A}, \mu)$ be a measure space. A subset A of X is called **a null set or a negligible set** if A is contained in a measurable subset of measure zero.

Remark

Let $(X, \mathscr{A})$ be a measurable space such that $\forall x \in X$; $\{x\} \in \mathscr{A}$. If we take $\mu = \delta_a$, with $a \in X$; then any subset $A \in \mathscr{A}$ such that $a \notin A$ is a null set.

Remarks

We denote by $\mathscr{N}$ the set of null sets. We have the following

1. $\emptyset \in \mathscr{N}$.
2. Any subset of a null set is a null set. If $A \subset B$ and $B \in \mathscr{N}$, then there exists $C \in \mathscr{B}$ such that $\mu(C) = 0$ and $B \subset C$; so $A \subset C$.
3. A countable union of null sets is a null set. If $(A_n)_n$ is any sequence in $\mathscr{N}$. For each $n \in \mathbb{N}$ choose $B_n \in \mathscr{B}$ such that $A_n \subset B_n$ and $\mu(B_n) = 0$. Now $B = \bigcup_{n=1}^{+\infty} B_n \in \mathscr{B}$ and $\bigcup_{n=1}^{+\infty} A_n \subset \bigcup_{n=1}^{+\infty} B_n$, and $\mu(\bigcup_{n=1}^{+\infty} B_n) \leq \sum_{n=0}^{+\infty} \mu(B_n)$, so $\mu(\bigcup_{n=1}^{+\infty} B_n) = 0$.

Definition

If $P(x)$ is some assertion applicable to numbers x of the set X, we say that

$$P(x) \text{ for almost every } x \in X \quad \text{or } P(x) \text{ a.e. } (x)$$

or

$$P(x) \text{ for } \mu - \text{almost every } x, \quad P(x)\,\mu - \text{a.e.}(x),$$

to mean that

$$\{x \in X;\ P(x) \ \text{ is false}\}$$

is a null set.

Definition

A measure space $(X, \mathscr{A}, \mu)$ is said to be complete if any null set is measurable ($\mathscr{N} \subset \mathscr{A}$), we say that the measure μ is complete.

Theorem

Let $(X, \mathscr{A}, \mu)$ be a measure space, and let $\mathscr{N}$ be the set of null subsets of X. Let $\mathscr{B} = \{A \cup B;\ A \in \mathscr{A} \text{ and } B \in \mathscr{N}\}$. $\mathscr{B}$ is a σ-algebra on X and there exists a unique measure ν which extends the measure μ on the σ-algebra $\mathscr{B}$. The measure space $(X, \mathscr{B}, \nu)$ is complete.

Proof

$\mathscr{B}$ is evidently closed under countable union. It suffices to prove that it is closed under complementarity. Let $A' = A \cup N$ be an element of $\mathscr{B}$. As N is a null set there exists B in $\mathscr{A} \cap \mathscr{N}$ and $N \subset B$. We have

$$A'^c = (A \cup N)^c = (A \cup B)^c \cup (B \setminus (A \cup N)).$$

It follows that A'^c is an element of $\mathscr{B}$.
If the measure ν exists it is unique. Indeed we must have $\nu(N) = 0$ for any $N \in \mathscr{N}$, thus if $A' = A \cup N$ is an element of $\mathscr{B}$ we shall have $\nu(A') = \mu(A)$.

To show that ν is a mapping on $\mathscr{B}$, we must show that if $A_1 \cup N_1 = A_2 \cup N_2$ with $A_1, A_2 \in \mathscr{A}$ and $N_1, N_2 \in \mathscr{N}$, then $\mu(A_1) = \mu(A_2)$. So we have $A_1 \setminus A_2 \subset N_2$, then it is a null set. If $B = A_1 \cap A_2$, then $A_1 = B \cup (A_1 \setminus A_2)$ and $\mu(B) = \mu(A_1)$. In the same way we have $\mu(B) = \mu(A_2)$, then $\mu(A_1) = \mu(A_2)$.
Let prove now that ν defines a measure on the σ-algebra $\mathscr{B}$. If $(A'_n)_n$ is a disjoint sequence in $\mathscr{B}$, with $A'_n = A_n \cup N_n$, $A_n \in \mathscr{A}$ and $N_n \in \mathcal{N}$; $\forall n \in \mathbb{N}$, we have

$$\nu(\bigcup_{n=1}^{+\infty} A'_n) = \nu\Big((\bigcup_{n=1}^{+\infty} A_n)\cup(\bigcup_{n=1}^{+\infty} N_n)\Big) = \mu(\bigcup_{n=1}^{+\infty} A_n) = \sum_{n=1}^{+\infty}\mu(A_n) = \sum_{n=1}^{+\infty}\nu(A'_n$$

Finally the measure space $(X, \mathscr{B}, \nu)$ is complete because the ν-null sets are elements of $\mathscr{N}$. It is evident that ν is the smallest complete extension of the measure μ.

□

Outer Measure

Definition

Let X be a non empty set. An outer measure or an exterior measure μ^* on X is a function $\mu^* \colon \mathscr{P}(X) \longrightarrow [0, \infty]$ which satisfies the following conditions

i) $\mu^*(\emptyset) = 0$.

ii) If $(A_n)_n$ is a sequence of subsets of X, then

$$\mu^*(\bigcup_{n=1}^{+\infty} A_n) \leq \sum_{n=1}^{+\infty} \mu^*(A_n).$$

iii) μ^* is increasing (i.e. $\mu^*(A) \leq \mu^*(B)$ if $A \subset B$).

Remark

Any measure on $\mathscr{P}(X)$ is an outer measure.

Definition

Let X be a set and μ^* be an outer measure on X. A subset A of X is called μ^*-measurable if

$$\forall B \subset X; \quad \mu^*(B) = \mu^*(B \cap A) + \mu^*(B \cap A^c). \qquad (2)$$

(The condition (2) is called the Caratheodory criterion.)

We introduce now the most important method of constructing measures called the Caratheodory's construction.

Theorem

[Caratheodory's Construction]
Let X be a non empty set and μ^* be an outer measure on X. Then the set $\mathscr{B}$ of μ^*-measurable subsets is a σ-algebra on X and the restriction of μ^* on $\mathscr{B}$ denoted $\mu = \mu^*|_{\mathscr{B}}$ is a complete measure.

Proof

i) $\emptyset$ is μ^*-measurable since $\mu^*(B\cap\emptyset)+\mu^*(B\cap\emptyset^c) = \mu^*(\emptyset)+\mu^*(B) = \mu^*(B)$.
ii) Let A be a μ^*-measurable set and B a subset of X. It follows from the definition of the outer measure that $\mu^*(B) = \mu^*(B\cap A) + \mu^*(B\cap A^c)$, then A^c is also μ^*-measurable.
iii) Let $A, B \in \mathscr{B}$ and E a subset of X. As A is measurable,

$$\begin{aligned} \mu^*(E\cap(A\cup B)) &= \mu^*(E\cap(A\cup B)\cap A) + \mu^*(E\cap(A\cup B)\cap A^c) \\ &= \mu^*(E\cap A) + \mu^*(E\cap B\cap A^c) \qquad (3) \end{aligned}$$

In use of the identity (3) and $A, B \in \mathscr{B}$, we have

$$\begin{aligned}\mu^*(E \cap (A \cup B)) + \mu^*(E \cap (A \cup B)^c) &= \mu^*(E \cap A) + \mu^*(E \cap B \cap A^c) + \mu^* \\ &= \mu^*(E \cap A) + \mu^*(E \cap A^c) = \mu^*(\end{aligned}$$

Then $A \cup B$ is measurable.
iv) Let A_1, A_2 be two disjoint measurable sets, B a subset of X and $E = B \cap (A_1 \cup A_2)$. Since $E \cap (A_1 \cup A_2)^c = \emptyset$, we have

$$\begin{aligned}\mu^*(E) = \mu^*(E \cap (A_1 \cup A_2)) + \mu^*(E \cap (A_1 \cup A_2)^c) &= \mu^*(E \cap A_1) + \mu^* \\ &= \mu^*(B \cap A_1) + \mu^*\end{aligned}$$

Thus

$$\mu^*(B \cap (A_1 \cup A_2)) = \mu^*(B \cap A_1) + \mu^*(B \cap A_2).$$

Let $(A_n)_n$ be a disjoint sequence in $\mathscr{B}$ and $B \subset X$, we have

$$\begin{aligned}
\mu^*(B) &= \mu^*(B \cap \bigcup_{j=1}^{n} A_j) + \mu^*(B \cap (\bigcup_{j=1}^{n} A_j)^c) \\
&\geq \mu^*(B \cap \bigcup_{j=1}^{n} A_j) + \mu^*(B \cap (\bigcup_{j=1}^{+\infty} A_j)^c) \\
&\geq \sum_{j=1}^{n} \mu^*(B \cap A_j) + \mu^*(B \cap (\bigcup_{j=1}^{+\infty} A_j)^c).
\end{aligned}$$

Then

$$\begin{aligned}\mu^*(B) &\geq \sum_{n=1}^{+\infty}\mu^*(B\cap A_n)+\mu^*(B\cap(\bigcup_{n=1}^{+\infty}A_n)^c) \qquad (4)\\ &\geq \mu^*(B\cap\bigcup_{n=1}^{+\infty}A_n)+\mu^*(B\cap(\bigcup_{n=1}^{+\infty}A_n)^c).\end{aligned}$$

The converse inequality results from the property of the outer measure μ^*.

To finish the proof we take a sequence $(B_n)_n$ in $\mathscr{B}$ and set $A_1 = B_1$, $A_n = B_n \setminus \bigcup_{j=1}^{n-1} B_j$. We have $\bigcup_{n=1}^{+\infty} A_n = \bigcup_{n=1}^{+\infty} B_n$. Thus $\mathscr{B}$ is a σ-algebra.

The restriction of μ^* on $\mathscr{B}$ is a measure is deduced from (4).

It remains to show that the measure μ^* is complete. To prove this, it suffices to prove that any null set A is measurable.

If A is a null set, then there exist $B \in \mathscr{B}$ such that $A \subset B$ and $\mu^*(B) = 0$. If E is a subset of X, $\mu^*(E \cap A) = 0$ and

$$\mu^*(E) \geq \mu^*(E \cap A^c) = \mu^*(E \cap A) + \mu^*(E \cap A^c).$$

The other inequality results from the definition of the outer measure μ^*. Thus A is μ^*-measurable.

□

Proposition

Let $(X, \mathscr{A}, \mu)$ be a measure space. We define the set function $\mu^* : \mathscr{P}(X) \longrightarrow [0, +\infty]$ by

$$\mu^*(A) = \inf\{\sum_{n=1}^{+\infty} \mu\ (A_n);\ A \subset \bigcup_{n=1}^{+\infty} A_n \text{ and } A_n \in \mathscr{A}\}. \tag{5}$$

μ^* is an outer measure and any measurable set is μ^*-measurable and the restriction of μ^* on $\mathscr{A}$ is equal to the measure μ.

Proof

It is easy to prove that $\mu^*(\emptyset) = 0$ and μ^* is increasing.
Let $(A_n)_n$ be a sequence of subsets of X. We claim that

$$\mu^*(\bigcup_{n=1}^{+\infty} A_n) \leq \sum_{n=1}^{+\infty} \mu^*(A_n).$$

If there exists a subset A_n such that $\mu^*(A_n) = +\infty$, then the inequality is trivial.
Assume now that $\forall n \in \mathbb{N}$; $\mu^*(A_n) < +\infty$.

For every $n \in \mathbb{N}$, and for every $\varepsilon > 0$, there exists a sequence $(A_{n,j})_j \in \mathscr{A}$, such that $\mu^*(A_n) \geq \sum_{j=1}^{+\infty} \mu(A_{n,j}) - \frac{\varepsilon}{2^n}$.
Then the sequence $(A_{n,j})_{j,n\in\mathbb{N}}$ is a covering of the set $A = \bigcup_{j=1}^{+\infty} A_n$ and $\sum_{n=1}^{+\infty}\sum_{j=1}^{+\infty} \mu(A_{n,j}) \leq \sum_{n=1}^{+\infty} \mu^*(A_n) + \varepsilon$.
Then $\mu^*(A) \leq \sum_{n=1}^{+\infty} \mu^*(A_n) + \varepsilon$, for all $\varepsilon > 0$ and thus $\mu^*(A) \leq \sum_{n=1}^{+\infty} \mu^*(A_n)$, which proves that μ^* is an outer measure.

Let now proving that $\mu^* = \mu$ on $\mathscr{A}$.
If $A \in \mathscr{A}$, then $\mu^*(A) \leq \mu(A)$, and if $\mu^*(A) = +\infty$ then $\mu^*(A) = \mu(A)$.
Assume now that $\mu^*(A) < +\infty$, then for every $\varepsilon > 0$, there exists a covering $(A_n)_n$ of A in $\mathscr{A}$ such that

$$\mu^*(A) \geq \sum_{n=1}^{+\infty} \mu(A_n) - \varepsilon.$$

Since $\mu(A) \leq \sum_{n=1}^{+\infty} \mu(A_n)$, then $\mu(A) \leq \mu^*(A) + \varepsilon$ for every $\varepsilon > 0$.
Thus $\mu(A) = \mu^*(A), \forall A \in \mathscr{A}$.

We claim to prove that any measurable set is μ^*-measurable.
By definition of the outer measure, if $A \in \mathscr{A}$ and $B \subset X$

$$\mu^*(B) \le \mu^*(B \cap A) + \mu^*(B \cap A^c).$$

If $\mu^*(B) = +\infty$, then $\mu^*(B) \ge \mu^*(B \cap A) + \mu^*(B \cap A^c)$.
Assume now that $\mu^*(B) < +\infty$, hence for $\varepsilon > 0$, there exists a covering $(B_n)_n$ of B in $\mathscr{A}$ such that $\mu^*(B) \ge \sum_{n=1}^{+\infty} \mu(B_n) - \varepsilon$. Since μ is a measure $\mu(A \cap B_n) + \mu(A^c \cap B_n) = \mu(B_n)$, then

$$\mu^*(B) \ge \sum_{n=1}^{+\infty} \mu(B_n \cap A) + \sum_{n=1}^{+\infty} \mu(B_n \cap A^c) - \varepsilon \ge \mu^*(B \cap A) + \mu^*(B \cap A^c) - \varepsilon.$$

We deduce that $\mu^*(B) \geq \mu^*(B \cap A) + \mu^*(B \cap A^c)$ and then $\mu^*(B) = \mu^*(B \cap A) + \mu^*(B \cap A^c)$. Which proves that A is μ^* measurable.

□

An outer measure can be also defined from any set function in the following sense

Proposition

Let $\mathcal{C} \subset \mathscr{P}(X)$ and $\rho\colon \mathcal{C} \longrightarrow [0, +\infty]$ be such that $\emptyset, X \in \mathcal{C}$ and $\rho(\emptyset) = 0$. For any $A \subset X$, define

$$\mu^*(A) = \inf\{\sum_{n=1}^{+\infty} \rho(A_n);\ A_n \in \mathcal{C} \text{ and } A \subset \cup_{n=1}^{+\infty} A_n\}. \qquad (6)$$

Then μ^* is an outer measure.

Proof

For any $A \subset X$ there exists $(A_n)_n$ in $\mathcal{C}$ such that $A \subset \cup_{n=1}^{+\infty} A_n$ (we can take $A_n = X$), then μ^* is well defined. Obviously $\mu^*(\emptyset) = 0$ and $\mu^*(A) \leq \mu^*(B)$ if $A \subset B$. To prove the countable subadditivity, let $(A_n)_n$ in $\mathscr{P}(X)$ and $A = \cup_{n=1}^{+\infty} A_n$. Without loss of generality, we can assume that $\rho(A_n) < +\infty$ for all $n \in \mathbb{N}$. For $\varepsilon > 0$ and for all $n \in \mathbb{N}$, there is a sequence $(A_{n,k})_k$ in $\mathcal{C}$ such that $A_n \subset \cup_{k=1}^{+\infty} A_{n,k}$ and $\displaystyle\sum_{k=1}^{+\infty} \rho(A_{n,k}) \leq \mu^*(A_n) + \frac{\varepsilon}{2^n}$. We have $A \subset \cup_{n,k=1}^{+\infty} A_{n,k}$ and

$$\sum_{n,k=1}^{+\infty} \rho(A_{n,k}) \leq \sum_{n=1}^{+\infty} \mu^*(A_n) + \varepsilon.$$

$\square$

Theorem

Let $(X, \mathscr{A}, \mu)$ be a $\sigma-$finite measure space and let μ^* be the outer measure defined by

$$\mu^*(A) = \inf\{\sum_{n=1}^{+\infty} \mu\ (A_n);\ A \subset \bigcup_{n=1}^{+\infty} A_n \text{ and } A_n \in \mathscr{A}\}.$$

We denote $\mathscr{B}$ the complete σ-algebra and $\mathscr{B}_0$ the σ-algebra of the μ^*-measurable sets. Then $\mathscr{B} = \mathscr{B}_0$.

Proof

According to the Proposition (45) $\mathscr{A} \subset \mathscr{B}_0$.
Let A be a null set, there exists a measurable set B such that $A \subset B$ and $\mu(B) = 0$. Let E be a subset of X;
$\mu^*(E \cap A) \leq \mu(B) = 0$ and $\mu^*(E \cap A^c) \leq \mu^*(E)$, then

$$\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c)$$

and $\mathscr{B} \subset \mathscr{B}_0$.

Let $A \in \mathscr{B}_0$, assume that $\mu^*(A) < +\infty$, then for all $n \in \mathbb{N}$, there exists a covering $(A_{j,n})_j \in \mathscr{A}$ of A such that

$$\sum_{j=1}^{+\infty} \mu(A_{j,n}) \leq \mu^*(A) + \frac{1}{n}.$$

We denote $B_n = \bigcup_{j=1}^{+\infty} A_{j,n}$. $A \subset B_n$ and $\mu(B_n) \leq \mu^*(A) + \frac{1}{n}$. Let $B = \bigcap_{n=1}^{+\infty} B_n$, $B \in \mathscr{A}$. Since $A \subset B$, $\mu^*(A) \leq \mu(B)$. Moreover $\mu(B) \leq \mu(B_n) \leq \mu^*(A) + \frac{1}{n}, \forall n \in \mathbb{N}$. Thus $\mu(B) = \mu^*(A)$.

Since $\mu^*(A) < \infty$, $\mu^*(B \setminus A) = 0$. Then $A = B \cap (B \setminus A)^c$ and $A^c = B^c \cup (B \setminus A)$.
$(B \setminus A)$ is a null set then it is in the σ-algebra $\mathscr{B}$. B is also in the σ-algebra $\mathscr{B}$. Then $A^c \in \mathscr{B}$ and also A.

If $\mu^*(A) = +\infty$. As μ is σ-finite, there exists a sequence $(E_n)_n$ of measurable sets such that $\mu(E_n) < +\infty$ and $\bigcup_{n=1}^{+\infty} E_n = X$. Then any $A \in \mathscr{B}_0$ is written as

$$A = \bigcup_{n=1}^{+\infty} A_n, \quad A_n \in \mathscr{B}_0, \textit{ and } \mu^*(A_n) < +\infty.$$

Then $A_n \in \mathscr{B}$ and $A \in \mathscr{B}$.

□

Extension of Measures

Theorem

Let μ_1 and μ_2 be two measures on a measurable space $(X, \mathscr{B})$. Assume that there exists a class $\mathscr{C}$ of measurable subsets such that
a) $\mathscr{C}$ is closed under finite intersection and that the σ-algebra generated by $\mathscr{C}$ is equal to $\mathscr{B}$.
b) There exists an increasing sequence $(E_n)_n$ in $\mathscr{C}$ such that $\lim\limits_{n\to+\infty} E_n = X$.
c) $\mu_1(C) = \mu_2(C) < +\infty$, for any $C \in \mathscr{C}$.
Then $\mu_1 = \mu_2$.

Proof

We suppose in the first case that $\mu_1(X) = \mu_2(X) < +\infty$.
Let $\mathscr{A} = \{A \in \mathscr{B};\ \mu_1(A) = \mu_2(A)\}$. By hypothesis $X \in \mathscr{C}$ and $\mathscr{C} \subset \mathscr{A}$. It is easy to prove that $\mathscr{A}$ is a monotone class. (If $(A_n)_n$ is an increasing sequence of $\mathscr{A}$, then $\mu_1(A_n) = \mu_2(A_n)$ for all n, and then

$$\mu_1(\bigcup_{n=1}^{+\infty} A_n) = \mu_2(\bigcup_{n=1}^{+\infty} A_n) = \mu_1(\lim_{n\to+\infty} A_n) = \mu_2(\lim_{n\to+\infty} A_n).$$

If $(A_n)_n$ is a decreasing sequence of $\mathscr{A}$, then $\mu_1(A_n) = \mu_2(A_n)$ for all n, as $\mu_1(X) = \mu_2(X) < +\infty$, then $\mu_1(\bigcap_{n=1}^{+\infty} A_n) = \mu_2(\bigcap_{n=1}^{+\infty} A_n)$.) $\mathscr{A}$ is a $\sigma-$algebra. (If $A, B \in \mathscr{A}$ with $A \subset B$, then $\mu_1(B \setminus A) = \mu_1(B) - \mu_1(A) = \mu_2(B) - \mu_2(A) = \mu_2(B \setminus A)$ and so $B \setminus A \in \mathscr{A}$. We use the fact that μ_1, μ_2 are finite and $\mu_1(X) = \mu_2(X)$). Then $\sigma(\mathscr{C}) = \mathscr{B} \subset \mathscr{A}$ and $\mathscr{A} = \mathscr{B}$ and $\mu_1 = \mu_2$.

In the general case we take $\mu_{j,n}$ the restriction of μ_j on E_n for all $n \in \mathbb{N}$. From the first case $\mu_{1,n} = \mu_{2,n}$, which gives $\mu_1 = \mu_2$, because $\mu_j = \lim_{n \to +\infty} \mu_{j,n}$; $j = 1, 2$.

□

Definition

If $\mathcal{A} \subset \mathscr{P}(X)$ is an algebra and $\mu \colon \mathcal{A} \longrightarrow [0, +\infty]$ is called a pre-measure if

a) $\mu(\emptyset) = 0$

b) If $(A_n) \in \mathscr{A}$ is a disjoint sequence, then

$$\mu(\cup_{n=1}^{+\infty} A_n) = \sum_{n=1}^{+\infty} \mu(A_n).$$

Remark

If μ is a pre-measure on an algebra $\mathcal{A} \subset P(X)$, it induces an outer measure on X defined by (6).

Proposition

If μ is a pre-measure on an algebra $\mathcal{A}$ and μ^* is defined by (6), then
a) $\mu^*_{\restriction \mathcal{A}} = \mu$,
b) every set in $\mathcal{A}$ is μ^* measurable.

Proof

a) Suppose $A \in \mathcal{A}$ and $A \subset \cup_{n=1}^{+\infty} A_n$, $A_n \in \mathcal{A}$ for all $n \in \mathbb{N}$. Let $B_n = A \cap (A_n \setminus \cup_{k=1}^{n-1} A_k)$. Then the sequence $(B_n)_n$ is disjoint in $\mathcal{A}$ whose union is A (i.e $(B_n)_n$ is a partition of A), so $\mu(A) = \sum_{n=1}^{+\infty} \mu(B_n)$.

It follows that $\mu(A) \leq \mu^*(A)$, and the reverse inequality is obvious since $A \subset \cup_{n=1}^{+\infty} A_n$, where $A_1 = A$ and $A_n = \emptyset$ for all $n \geq 2$.

b) If $A \in \mathcal{A}$, $B \subset X$, and $\varepsilon > 0$, there is a sequence $(B_n)_n \in \mathcal{A}$ with $B \subset \cup_{n=1}^{+\infty} B_n$ and $\sum_{n=1}^{+\infty} \mu(B_n) \leq \mu^*(B) + \varepsilon$. Then

$$\mu^*(B \cap A) + \mu^*(B \cap A^c) \leq \sum_{n=1}^{+\infty} \mu(B_n \cap A) + \sum_{n=1}^{+\infty} \mu(B_n \cap A^c) \leq \mu^*(B) + \varepsilon.$$

Since ε is arbitrary, A is μ^* measurable. □

Theorem

Let $\mathcal{A} \subset \mathscr{P}(X)$ be an algebra, μ_1 a pre-measure on $\mathcal{A}$, and $\mathscr{A}$ the $\sigma-$algebra generated by $\mathcal{A}$. There exists a measure μ on $\mathscr{A}$ whose restriction to $\mathcal{A}$ is μ_1, $\mu = {\mu_1}^*_{\restriction \mathscr{A}}$ where μ_1^* is defined by (6) (relatively to μ_1). If ν is another measure on $\mathscr{A}$ that extends μ_1, then $\nu(A) \leq \mu(A)$ for all $A \in \mathscr{A}$, with equality when $\mu(A) < \infty$. If μ_1 is $\sigma-$finite, then μ is the unique extension of μ to a measure on $\mathscr{A}$.

Proof

The first assertion follows from Caratheodory's theorem and Proposition (62) since the $\sigma-$algebra of μ^*-measurable sets includes $\mathcal{A}$ and hence includes $\mathscr{A}$. For the second assertion, if $A \in \mathscr{A}$and $A \subset \cup_{n=1}^{+\infty} A_n$, where $A_n \in \mathcal{A}$, $\nu(A) \leq \sum_{n=1}^{+\infty} \mu_1(A_n) = \sum_{n=1}^{+\infty} \mu(A_n)$, hence $\nu(A) \leq \mu^*(A) = \mu(A)$. Also, if $B = \cup_{n=1}^{+\infty} A_n$, we have

$$\nu(B) = \lim_{n\to+\infty} \nu(\cup_{k=1}^{n} A_k) = \lim_{n\to+\infty} \mu(\cup_{k=1}^{n} A_k) = \mu(B).$$

If $\mu(A) < +\infty$, we can choose the sequence $(A_n)_n$ so that $\mu(B) \leq \mu(A) + \varepsilon$, hence $\mu(B \setminus A) \leq \varepsilon$ and

$$\mu(A) \leq \mu(B) = \nu(B) = \nu(A)+\nu(B\setminus A) \leq \nu(A)+\mu(B\setminus A) \leq \nu(A)+\varepsilon.$$

Since ε is arbitrary, $\mu(A) = \nu(A)$.
Finally, suppose $X = \cup_{n=1}^{+\infty} K_n$ with $\mu(K_n) < +\infty$, where we can assume that the K_n are disjoint. Then for any $A \in \mathscr{A}$,

$$\mu(A) = \sum_{n=1}^{+\infty} \mu(A \cap K_n) = \sum_{n=1}^{+\infty} \nu(A \cap K_n) = \nu(A),$$

so $\nu = \mu$. □

Lebesgue Measure on $\mathbb{R}$

Theorem

There exists only and only one measure λ on $\mathscr{B}_{\mathbb{R}}$ satisfying
i) λ invariant under translation. (i.e.
$\forall x \in \mathbb{R},\ \forall A \in \mathscr{B}_{\mathbb{R}};\ \lambda(x + A) = \lambda(A)$).
ii) $\lambda([0, 1]) = 1$.

Proof

Uniqueness

Assume that there exists two measures μ and ν on $\mathscr{B}_{\mathbb{R}}$ satisfying (i) and (ii) then $\nu[0, \frac{1}{n}[\leq \frac{1}{n} \ \Rightarrow \nu\{0\} = 0$ and then any finite set or countable set is a null set and all the intervals $[a, b]$, $]a, b]$, $[a, b[$ and $]a, b[$ have the same measure and equal to $b - a$. (We treat the case of a and b rational and take the limit.)
We denote by $\mathscr{E}$ the set of finite union of intervals of $\mathbb{R}$ in the form $[a, b[; \ a, b \in \mathbb{R}$. The set $\mathscr{E}$ is closed under finite intersection and $\mathbb{R} = \bigcup_n [-n, n[$. Then $\mu = \nu$ on $\mathscr{E}$. It follows from the uniqueness theorem (5) that μ and ν are equal on $\mathscr{B}_{\mathbb{R}}$.
Existence We need the following lemma.

Lemma

Define for any subset A of $\mathbb{R}$

$$\mu^*(A) = \inf_{\mathscr{R}} \sum_{I \in \mathscr{R}} \mathscr{L}(I).$$

$\mathscr{R}$ describes the whole of finite or countable coverings of A by open intervals, and $\mathscr{L}(I)$ the length of I. μ^* fulfills the following properties.

1. μ^* is an outer measure. We denote $\mathscr{B}^*_{\mathbb{R}}$ the σ-algebra of μ^*-measurable sets.
2. For all interval I of $\mathbb{R}$, $\mu^*(I) = \mathscr{L}(I)$.

Lemma

3. If Ω is an open set of $\mathbb{R}$ and $(I_n)_n$ its connected components, then $\mu^*(\Omega) = \sum_{n=1}^{+\infty} \mathscr{L}(I_n)$.
4. For any subset $A \subset \mathbb{R}$,

$$\mu^*(A) = \inf_{O \text{ open} \supset A} \mu^*(O).$$

Proof

1. By proposition (51) μ^* is an outer measure.
2. If a and b are the endpoints of I and $\varepsilon > 0$, then $I \subset]a-\varepsilon, b+\varepsilon[$ and $\mu^*(I) \leq \mathscr{L}(I) + 2\varepsilon$. It follows that $\mu^*(I) \leq \mathscr{L}(I)$.
   Conversely let $(I_k)_k$ be an open covering of I, then $[a+\varepsilon, b-\varepsilon] \subset \cup_{k=1}^{+\infty} I_k$. As $[a+\varepsilon, b-\varepsilon]$ is compact, there exist a finite sub-covering $(I_k)_{1\leq k\leq n}$ such that $[a+\varepsilon, b-\varepsilon] \subset \cup_{k=1}^{n} I_k$. It results that
   $$b-a-2\varepsilon \leq \sum_{k=1}^{n} \mathscr{L}(I_k) \leq \sum_{k=1}^{+\infty} \mathscr{L}(I_k).$$ Thus
   $b-a-2\varepsilon \leq \mu^*(I)$ for any $\varepsilon > 0$, hence $\mathscr{L}(I) = \mu^*(I)$.

3. Let Ω be an open set of $\mathbb{R}$ and $(I_n)_n$ its connected components, from the definition of μ^*

$$\mu^*(\Omega) \leq \sum_{n=1}^{+\infty} \mathscr{L}(I_n). \tag{7}$$

Conversely let $(J_k)_k$ be a covering of Ω by open intervals, we have $I_n = \bigcup_{k=1}^{+\infty} J_k \cap I_n$. It results that

$$\sum_{n=1}^{+\infty} \mathscr{L}(I_n) \leq \sum_{n=1}^{+\infty} \sum_{k=1}^{+\infty} \mathscr{L}(I_n \cap J_k) = \sum_{k=1}^{+\infty} \sum_{n=1}^{+\infty} \mathscr{L}(I_n \cap J_k).$$

In the other hand the intervals $(I_n)_n$ are disjoints, then for any m, $\displaystyle\bigcup_{n=1}^{m}(J_k \cap I_n) \subset J_k$ and for all $m \in \mathbb{N}$; $\displaystyle\sum_{n=1}^{m} \mathscr{L}(J_k \cap I_n) \leq \mathscr{L}(J_k)$. It results that $\displaystyle\sum_{n=1}^{+\infty} \mathscr{L}(I_n \cap J_k) \leq \sum_{k=1}^{+\infty} \mathscr{L}(J_k)$.
Then

$$\sum_{n=1}^{+\infty} \mathscr{L}(I_n) \leq \mu^*(\Omega). \tag{8}$$

So the relations (7) and (8) gives that $\mu^*(\Omega) = \sum_{n=1}^{+\infty} \mathscr{L}(I_n)$.

4. Let $A \subset \mathbb{R}$ and $(I_n)_n$ a countable covering of A by open intervals. If $\omega = \bigcup_{n=1}^{+\infty} I_n$, $\mu^*(A) \leq \mu^*(\omega) \leq \sum_{n=1}^{+\infty} \mathscr{L}(I_n)$. Then $\mu^*(A) \leq \inf_{O \text{ open} \supset A} \mu^*(O)$. The converse inequality is evident if $\mu^*(A) = +\infty$.
If $\mu^*(A) < +\infty$, for $\varepsilon > 0$, there exists a countable covering $(I_n)_n$ of A by open intervals such that $\sum_{n=1}^{+\infty} \mathscr{L}(I_n) \leq \mu^*(A) + \varepsilon$.
The open set $\Omega = \cup_{n=1}^{+\infty} I_n$ contains A and
$\mu^*(O) \leq \sum_{n=1}^{+\infty} \mathscr{L}(I_n) \leq \mu^*(A) + \varepsilon$. Then
$\inf_{O \text{ open} \supset A} \mu^*(O) \leq \mu^*(A)$.

□

Remark

The previous lemma proves also easily that μ^* is an outer measure on $\mathscr{P}(\mathbb{R})$. Indeed
i) $\mu^*(\emptyset) = 0$.
ii) If $A \subset B$, then $\mu^*(A) = \inf_{\omega(open)\supset A} \mu^*(\omega) \leq \inf_{\omega(open)\supset B} \mu^*(\omega) = \mu^*(B)$.
iii) If $(A_n)_n$ is a sequence of subsets of $\mathbb{R}$. It suffices to prove that

$$\mu^*(\cup_{n=1}^{+\infty} A_n) \leq \sum_{n=1}^{+\infty} \mu^*(A_n). \tag{9}$$

If there exists n_0 such that $\mu^*(A_{n_0}) = +\infty$, the inequality (9) is trivially fulfilled.
If $\mu^*(A_n) < +\infty$ for all $n \in \mathbb{N}$, for $\varepsilon > 0$, then for any $n \in \mathbb{N}$ there exists an open set ω_n containing A_n such that $\mu^*(\omega_n) \leq \mu^*(A_n) + \frac{\varepsilon}{2^n}$.

$$\mu^*(\cup_{n=1}^{+\infty} A_n) \leq \mu^*(\cup_{n=1}^{+\infty} \omega_n) \leq \sum_{n=1}^{+\infty} \mu^*(\omega_n) \leq \sum_{n=1}^{+\infty} \mu^*(A_n) + \sum_{n=1}^{+\infty} \frac{\varepsilon}{2^n} = \sum_{n=1}^{+\infty} \mu^* \tag{10}$$

for any $\varepsilon > 0$, thus $\mu^*(\bigcup_{n=1}^{+\infty} A_n) \leq \sum_{n=1}^{+\infty} \mu^*(A_n)$.

Proposition

Any Borelian subset is Lebesgue measurable i.e $\mathscr{B}_{\mathbb{R}} \subset \mathscr{B}^*_{\mathbb{R}}$.

Proof

It suffices to prove that $\forall a \in \mathbb{R}$, $]a, +\infty[\in \mathscr{B}^*_{\mathbb{R}}$. Let E be a subset of $\mathbb{R}$, our goal is to prove that

$$\mu^*(E) = \mu^*(E \cap]a, +\infty[) + \mu^*(E \cap] - \infty, a]). \tag{11}$$

Since μ^* is an outer measure, $\mu^*(E) \leq \mu^*(E \cap]a, +\infty[) + \mu^*(E \cap] - \infty, a])$. For the converse inequality the result is trivial if $\mu^*(E) = +\infty$. Assume that $\mu^*(E) < +\infty$. Let $\varepsilon > 0$ there exists an open set $\Omega_\varepsilon \supset E$ such that : $\mu^*(\Omega_\varepsilon) \leq \mu^*(E) + \varepsilon$. Assume in the first time that $a \notin \Omega_\varepsilon$.

$$\mu^*(\Omega_\varepsilon) = \sum_{I\in\mathcal{C}} \mathscr{L}(I) = \sum_{I\in\mathcal{C}\cap]a,+\infty[} \mathscr{L}(I) + \sum_{I\in\mathcal{C}\cap]-\infty,a[} \mathscr{L}(I) \quad (12)$$

with $\mathcal{C}$ the set of the connected components of Ω_ε. Then

$$\mu^*(\Omega_\varepsilon) = \mu^*(\Omega_\varepsilon\cap]a,+\infty[)+\mu^*(\Omega_\varepsilon\cap]-\infty,a[) \geq \mu^*(E\cap]a,+\infty[)+\mu^*(E\cap]$$

Therefore $\mu^*(E) \geq \mu^*(E\cap]a,+\infty[) + \mu^*(E\cap]-\infty,a])$.

If now $a \in \Omega_\varepsilon$, let $\Omega'_\varepsilon = \Omega_\varepsilon \setminus \{a\}$. According to the first remark $\mu^*(\Omega'_\varepsilon) = \mu^*(\Omega_\varepsilon)$. □

Proof

of theorem (6)
According to the theorem (37), $\mathscr{B}^*_{\mathbb{R}}$ is a σ-algebra and $\lambda = \mu^*|_{\mathscr{B}^*_{\mathbb{R}}}$ is a complete measure. $\mathscr{B}^*_{\mathbb{R}}$ is called the **Lebesgue** σ**-algebra** and the elements of $\mathscr{B}^*_{\mathbb{R}}$ are called the **Lebesgue measurable sets**. The measure λ is complete and called the **Lebesgue measure** on $\mathbb{R}$. λ fulfills the theorem (6). $\square$

Proposition

Let $\mathscr{B}^*_{\mathbb{R}}$ be the Lebesgue σ-algebra on $\mathbb{R}$, then $\forall\ A \in \mathscr{B}^*_{\mathbb{R}}$

$$\lambda(A) = \inf_{\omega \ \ open \supset A} \lambda(\omega)$$

$$\lambda(A) = \sup_{K \ \ compact \subset A} \lambda(K).$$

We say that the measure λ is regular.

Proof

If A is bounded, there exists $n \in \mathbb{N}$ such that $A \subset [-n, n]$. For any $\varepsilon > 0$, the set $[-n, n] \setminus A$ is measurable, then there exists an open set $\omega \supset ([-n, n] \setminus A)$ such that

$$\lambda(\omega) \leq \lambda([-n, n] \setminus A) + \varepsilon = \lambda[-n, n] - \lambda(A) + \varepsilon$$

because $\lambda([-n, n] \setminus A) = \inf_{\omega \ open \supset ([-n,n] \setminus A)} \lambda(\omega)$.

Let $K = [-n, n] \cap \omega^c$. K is a compact in A.

$$2n = \lambda[-n, n] = \lambda([-n, n]\cap\omega^c)+\lambda([-n, n]\cap\omega) \leq \lambda(K)+\varepsilon+\lambda[-n, n]-\lambda(A$$

Then $\lambda(A) \leq \lambda(K) + \varepsilon$ and $\lambda(A) = \sup_{K\ compact\subset A} \lambda(K)$.
If A is not bounded, then $\forall n \in \mathbb{N}$ there exists a compact $K_n \subset [-n, n] \cap A$ such that

$$\lambda(K_n) \geq \lambda([-n, n] \cap A) - \frac{1}{n},$$

then

$$\sup_{K\ compact\subset A} \lambda(K) \geq \sup_n(\lambda(K_n)) \geq \lim_{n\to+\infty}(\lambda([-n, n]\cap A)-\frac{1}{n}) = \lambda(A)$$

□

Lebesgue Measure on $\mathbb{R}^n$

Lebesgue Outer Measure on $\mathbb{R}^n$

Definition

1. A subset R of $\mathbb{R}^n$ is called a rectangle if $R = \prod_{j=1}^{n} I_j$, where I_j are intervals of $\mathbb{R}$.
   The rectangle R is called an open rectangle (resp closed rectangle) if the intervals $(I_j)_j$ are open (resp closed).

Definition

2. If R is a closed rectangle defined by $R = \{x \in \mathbb{R}^n; a_j \leq x_j \leq b_j,\ 1 \leq j \leq n\}$, we set $\mathrm{Vol}(R) = \prod_{j=1}^{n}(b_j - a_j)$ called the volume of R. If the rectangle is not closed we take the same volume that the one of its closure.

Definition

Let $A \subset \mathbb{R}^n$. We say that a family of rectangles $(R_j)_{j\in I}$ is a covering of A if $A \subset \cup_{j\in I} R_j$. If each of the rectangle R_j in the covering is open (resp. closed), the covering is called an open (resp. closed) covering of A.

Remark

Any subset of $\mathbb{R}^n$ admits a countable covering of rectangles. (We can take the rectangle of rational sides and centered at points with rational coordinates.)

Exercise

If a rectangle $R \subset \cup_{j=1}^{m} R_j$, then $\mathrm{Vol}(R) \leq \sum_{j=1}^{m} \mathrm{Vol}(R_j)$.

Definition

A collection of rectangles $(R_j)_j$ is called almost disjoint if the interiors of R_j are pairwise disjoint (disjoint for simplicity).

Lemma

If a rectangle $R = \cup_{j=1}^m R_j$ such that R_j are pairwise almost disjoint, then $\mathrm{Vol}(R) = \sum_{j=1}^m \mathrm{Vol}(R_j)$.

Proof

It is obvious that $\mathrm{Vol}(R) \le \sum_{j=1}^m \mathrm{Vol}(R_j)$.

Moreover $\cup_{j=1}^m \overset{o}{R}_j \subset \overset{o}{R}$ and the union is disjoint. Then $\displaystyle\sum_{j=1}^m \mathrm{Vol}(R_j) \le \mathrm{Vol}(R)$ and then $\mathrm{Vol}(R) = \sum_{j=1}^m \mathrm{Vol}(R_j)$.

□

Remark

An open subset $\Omega \subset \mathbb{R}^n$, for $n \geq 2$ is not in general a countable union of pairwise disjoint rectangles. But in $\mathbb{R}$ we know that for every open subset $\Omega \subset \mathbb{R}$, there exists a unique countable family of open intervals I_j such that $\Omega = \cup_{n=1}^{+\infty} I_n$ where $(I_n)_n$ are pairwise disjoint. (The intervals I_n are the connected components of Ω).

Theorem

For every open subset $\Omega \subset \mathbb{R}^n$, there exists a countable family of almost disjoint closed rectangles $(R_n)_n$ such that $\Omega = \cup_{n=1}^{+\infty} R_n$.

Proof

We consider the closed rectangles in $\mathbb{R}^n$ of side length 1 and whose vertices have integer coordinates. The number of rectangles is countable and they are almost disjoint. We denote by $\mathcal{C}_1$ those which are contained in Ω.
We bisect the rectangles of the family $\mathcal{C}' = \mathcal{R}_1 \setminus \mathcal{C}_1$ which intersect both Ω and Ω^c in to 2^n rectangles of side length $\frac{1}{2}$. Again we consider the family $\mathcal{C}_2$ which are contained in Ω.
Repeating this procedure, we have a countable collection $(\mathcal{C}_k)_k$ of almost disjoint rectangles in Ω. By construction, $\cup_{R \in \cup_{k=1}^{+\infty} \mathcal{C}_k} R \subset \Omega$.
For $x \in \Omega$, there is a rectangle of side length $\frac{1}{2^m}$ in $\mathcal{C} = \cup_{k=1}^{+\infty} \mathcal{C}_k$ which contains x, thus, $\cup_{R \in \mathcal{C}} R = \Omega$.

□

Theorem and Definition

For any subset A of $\mathbb{R}^n$ we define the following set function $\lambda_n^*(A)$ as,

$$\lambda_n^*(A) = \inf \sum_{k=1}^{+\infty} \mathrm{Vol}(R_k),$$

the infimum being taken over all countable coverings of A by rectangles.
λ_n^* is an outer measure called the Lebesgue outer measure.

Lemma

The outer measure λ_n^* is unchanging if we restrict ourselves to open covering or closed covering, i.e., for every subset $A \subset \mathbb{R}^n$,

$$\lambda_n^*(A) = \inf \sum_{k=1}^{+\infty} \mathrm{Vol}(R_k),$$

where the infimum is taken over all possible countable closed or open coverings $(R_k)_k$ of A.

Proof

It suffices to solve the problem for bounded subsets. Let $A \subset \mathbb{R}^n$ be a bounded subset. For every $\varepsilon > 0$, there is a covering $(R_k)_k$ of A and $\sum_{k=1}^{+\infty} \mathrm{Vol}(R_k) \leq \lambda_n^*(A) + \varepsilon$. The sequence of closed rectangles $(\overline{R_k})_k$ is a covering of A and $\sum_{k=1}^{+\infty} \mathrm{Vol}(\overline{R_k}) = \sum_{k=1}^{+\infty} \mathrm{Vol}(R_k) \leq \lambda_n^*(A) + \varepsilon$.
If $(a_{j,k})_j$, $(b_{j,k})_j$, $j = 1 \dots n$ are the edges of the rectangle R_k, we define the rectangle $\tilde{R}_k = \prod_{j=1}^{n}]a_{j,k} - \delta_k, b_{j,k} + \delta_k[$ such that $\mathrm{Vol}(\tilde{R}_k) \leq \mathrm{Vol}(R_k) + \dfrac{\varepsilon}{2^k}$. (The rectangle R_k is not necessary open or closed).

The sequence of open rectangles $(\tilde{R}_k)_k$ is a covering of A and

$$\sum_{k=1}^{+\infty} \mathrm{Vol}(\tilde{R}_k) \leq \sum_{k=1}^{+\infty} \mathrm{Vol}(R_k) + \sum_{k=1}^{+\infty} \frac{\varepsilon}{2^k} \leq \lambda_n^*(A) + 2\varepsilon.$$

□

Lemma

If R is a rectangle in $\mathbb{R}^n$, then $\lambda_n^*(R) = \mathrm{Vol}(R)$.

Proof

It is evident that $\lambda_n^*(R) \leq \mathrm{Vol}(R)$. Conversely let $\varepsilon > 0$ and let $(R_k)_k$ be a covering of R by open rectangles. We denote by R_ε a rectangle obtained by dilation of the sides of R such that $\mathrm{Vol}(R) \leq \mathrm{Vol}(\overline{R_\varepsilon}) + \varepsilon$ and $\overline{R_\varepsilon} \subset R \subset \displaystyle\bigcup_{k=1}^{+\infty} R_k$. Since $\overline{R_\varepsilon}$ is compact, there exists a finite sub-covering $(R_k)_{1\leq k\leq m}$ of $\overline{R_\varepsilon}$. Thus $\mathrm{Vol}(\overline{R_\varepsilon}) \leq \displaystyle\sum_{k=1}^{m} \mathrm{Vol}(R_k) \leq \sum_{k=1}^{+\infty} \mathrm{Vol}(R_k)$. It results that $\mathrm{Vol}(R) - \varepsilon \leq \lambda_n^*(R)$, for all $\varepsilon > 0$. Then $\mathrm{Vol}(R) = \lambda_n^*(R)$.

□

Remark

For $n \geq 2$, $\lambda_n^*(\mathbb{R}^{n-1}) = 0$, where $\mathbb{R}^{n-1}$ considered as subset of $\mathbb{R}^n$. Choose a covering $(R_j)_j$ of $\mathbb{R}^{n-1}$ in $\mathbb{R}^{n-1}$ such that $\mathrm{Vol}(R_j)_{n-1} = 1$. Then $A_j = R_j \times]-\frac{\varepsilon}{2^j}, \frac{\varepsilon}{2^j}[$ is a covering of $\mathbb{R}^{n-1}$ in $\mathbb{R}^n$. Thus,

$$\lambda_n^*(\mathbb{R}^{n-1}) \leq \sum_{j=1}^{+\infty} \mathrm{Vol}(A_j) = \sum_{j=1}^{+\infty} \frac{2\varepsilon}{2^j} = 2\varepsilon.$$

Theorem

[Outer Regularity]
If $A \subset \mathbb{R}^n$ is a subset of $\mathbb{R}^n$, then $\lambda_n^*(A) = \inf_{\Omega \supset A} \lambda_n^*(\Omega)$, where Ω are open sets containing A.

Proof

By monotonicity, $\lambda_n^*(A) \leq \lambda_n^*(\Omega)$ for all open set Ω containing A. Thus, $\lambda_n^*(A) \leq \inf \lambda_n^*(\Omega)$.
Conversely, for $\varepsilon > 0$ there exists an open covering of rectangles $(R_j)_j$ of A such that $\displaystyle\sum_{j=1}^{+\infty} \mathrm{Vol}(R_j) \leq \lambda_n^*(A) + \varepsilon$. The open set $\Omega = \cup_{j=1}^{+\infty} R_j$ contains A and $\lambda_n^*(\Omega) \leq \sum_{j=1}^{+\infty} \mathrm{Vol}(R_j) \leq \lambda_n^*(A) + \varepsilon$.

$\square$

Definition

A subset A is said to be $\mathscr{G}_\delta$ if it is a countable intersection of open sets in $\mathbb{R}^n$. We say A is $\mathcal{F}_\sigma$ if it is a countable union of closed sets in $\mathbb{R}^n$.

Corollary

For every subset $A \subset \mathbb{R}^n$ there exists a $\mathscr{G}_\delta$ subset G of $\mathbb{R}^n$ such that $G \supset A$ and $\lambda_n^*(A) = \lambda_n^*(G)$.

Proof

By Theorem (101), for every $k \in \mathbb{N}$ there is an open set $\Omega_k \supset A$ such that $\lambda_n^*(\Omega_k) \leq \lambda_n^*(A) + \frac{1}{k}$. The set $G = \cap_{k=1}^{+\infty} \Omega_k$ is a $\mathscr{G}_\delta$ set and contains A and hence $\lambda_n^*(A) \leq \lambda_n^*(G)$. For the reverse inequality, we note that $G \subset \Omega_k$, for all k, and by monotonicity $\lambda_n^*(G) \leq \lambda_n^*(\Omega_k) \leq \lambda_n^*(A) + \frac{1}{k}$, for all k. Thus, $\lambda_n^*(G) = \lambda_n^*(A)$. $\square$

Definition

An outer measure μ^* on a metric space (X, d) is called a metric outer measure if for all $A, B \subset X$ such that $d(A, B) > 0$, then $\mu^*(A \cup B) = \mu^*(A) + \mu^*(B)$.

Proposition

If A and B are subsets of $\mathbb{R}^n$ such that $d(A, B) > 0$, then $\lambda_n^*(A \cup B) = \lambda_n^*(A) + \lambda_n^*(B)$. (i.e. λ_n^* is a metric outer measure).

Proof

We have $\lambda_n^*(A \cup B) \leq \lambda_n^*(A) + \lambda_n^*(B)$.

For $\varepsilon > 0$, there is a covering $(R_j)_j$ of $A \cup B$ such that $\sum_{j=1}^{+\infty} \mathrm{Vol}(R_j) \leq \lambda_n^*(A \cup B) + \varepsilon$.
Without loss of generality, we can assume that the diameter of R_j are less then $d(A, B)$. (For all $j \in \mathbb{N}$, there is a finite covering of R_j by rectangles of diameter less then $d(A, B)$. We have also $\sum_{j=1}^{+\infty} \mathrm{Vol}(R_j) \leq \lambda_n^*(A \cup B) + \varepsilon$).

It results that there no rectangle R_j which intersects A and B at the same time. Let $I = \{j;\ R_j \cap A \neq \emptyset\}$ and $J = \{j;\ R_j \cap B \neq \emptyset\}$. I and J are disjoint. Thus, $(R_j)_{j\in I}$ is an open covering of A and $(R_j)_{j\in J}$ is an open covering of B. Thus,

$$\lambda_n^*(A)+\lambda_n^*(B) \leq \sum_{j\in I} \mathrm{Vol}(R_j)+\sum_{j\in J} \mathrm{Vol}(R_j) \leq \sum_{j=1}^{+\infty} \mathrm{Vol}(R_j) \leq \lambda_n^*(A\cup B)+\varepsilon,$$

which proves that $\lambda_n^*(A) + \lambda_n^*(B) \leq \lambda_n^*(A \cup B)$. $\square$

Proposition

If $(R_j)_j$ is a sequence of almost disjoint closed rectangles, then

$$\lambda_n^*(\cup_{j=1}^{+\infty} R_j) = \sum_{j=1}^{+\infty} \mathrm{Vol}(R_j).$$

Proof

If $A = \cup_{j=1}^{+\infty} R_j$, then $\lambda_n^*(A) \leq \sum_{j=1}^{+\infty} \mathrm{Vol}(R_j)$.

For $\varepsilon > 0$, let $S_j \subset R_j$ be open rectangle such that
i) $\mathrm{Vol}(R_j) \leq \mathrm{Vol}(S_j) + \frac{\varepsilon}{2^j}$
ii) $d(S_j, S_k) > 0$, for all $j \neq k$.

Then by Proposition (105), $\lambda_n^*(\cup_{j=1}^{k} S_j) = \sum_{j=1}^{k} \mathrm{Vol}(S_j)$ for all $k \in \mathbb{N}$.

Since $\cup_{j=1}^{k} S_j \subset A$,

$$\lambda_n^*(A) \geq \lambda_n^*(\cup_{j=1}^{k} S_j) = \sum_{j=1}^{k} \mathrm{Vol}(S_j) \geq \sum_{j=1}^{k} \mathrm{Vol}(R_j) - \frac{\varepsilon}{2^j}.$$

Corollary

If Ω is an open set and $\Omega = \cup_{j=1}^{+\infty} R_j$, where $(R_j)_j$ is an almost disjoint closed rectangles, then

$$\lambda_n^*(\Omega) = \sum_{j=1}^{+\infty} \mathrm{Vol}(R_j).$$

Theorem

Let $\mathscr{B}^*_{\mathbb{R}^n}$ be the σ-algebra of λ^*_n-measurable sets. The restriction of λ^*_n on the σ-algebra $\mathscr{B}^*_{\mathbb{R}^n}$ is a measure called the **Lebesgue measure** on $\mathbb{R}^n$ and denoted by λ_n. The σ-algebra $\mathscr{B}^*_{\mathbb{R}^n}$ contains the Borel σ-algebra $\mathscr{B}_{\mathbb{R}^n}$. Moreover the measure λ_n is the unique measure invariant under translation on $\mathbb{R}^n$. Moreover the measure λ_n is regular.

Proof

Let $\Omega = \bigcup_{k=1}^{m} R_k$ be a finite union of disjoint rectangles, then $\lambda_n^*(\Omega) = \sum_{k=1}^{m} \mathrm{Vol}(R_k)$. Indeed it is evident that $\lambda_n^*(\bigcup_{k=1}^{m} R_k) \leq \sum_{k=1}^{m} \mathrm{Vol}(R_k)$ where the rectangles can be opens or not, it suffices to take the closure of R_k.

Conversely if $(Q_k)_k$ is a covering of Ω by rectangles, then $R_j = \cup_{k=1}^{+\infty}(Q_k \cap R_j) \Rightarrow \mathrm{Vol}(R_j) = \lambda_n^*(R_j) \leq \sum_k \mathrm{Vol}(Q_k \cap R_j)$ and since $(R_j)_j$ are disjoints; $\sum_{j=1}^{m} \mathrm{Vol}(Q_k \cap R_j) = \mathrm{Vol}(Q_k)$. It results that $\sum_{k=1}^{m} \mathrm{Vol}(R_k) \leq \lambda_n^*(\cup_{k=1}^{m} R_k)$, this which yields the result. The result remains valid if we have a sequence $(R_k)_k$ of disjoints rectangles and if $E = \bigcup_{k=1}^{+\infty} R_k$, then $\lambda_n^*(E) = \sum_k \mathrm{Vol}(R_k)$.

Let $A \in \mathscr{B}^*_{\mathbb{R}^n}$ and $(R_k)_k$ a covering of A by open rectangles. We set $\Omega = \bigcup_{k=1}^{+\infty} R_k \supset A$, thus $\lambda_n^*(A) \leq \lambda_n^*(\Omega) \leq \sum_{k=1}^{+\infty} \mathrm{Vol}(R_k)$. It results that

$$\lambda_n^*(A) = \inf_{\Omega \text{ open} \supset A} \lambda_n^*(\Omega).$$

Let prove now that the open rectangles are measurable with respect to the outer measure λ_n^*. Let E be a subset of $\mathbb{R}^n$ and R an open rectangle, we claim that

$$\lambda_n^*(R \cap E) + \lambda_n^*(E \cap R^c) = \lambda_n^*(E).$$

We have evidently $\lambda_n^*(E) \leq \lambda_n^*(R \cap E) + \lambda_n^*(E \cap R^c)$. For the other sense the result is trivial if $\lambda_n^*(E) = +\infty$. Assume now that $\lambda_n^*(E) < +\infty$. Let $\varepsilon > 0$, there exists an open set Ω_ε such that $\lambda_n^*(\Omega_\varepsilon) \leq \lambda_n^*(E)+\varepsilon$. Assume in the first time that the boundary of R denoted by ∂R is in the complementary of Ω_ε. Thus if $\Omega_\varepsilon = \bigcup_{k=1}^{+\infty} R_k$ is union of disjoint open rectangles, we have

$$\begin{aligned}\lambda_n^*(\Omega_\varepsilon) &= \sum_{k=1}^{+\infty}\mathrm{Vol}(R_k) = \sum_{R_k\subset R}\mathrm{Vol}(R_k) + \sum_{R_k\subset R^c}\mathrm{Vol}(R_k)\\ &= \lambda_n^*(\Omega_\varepsilon\cap R) + \lambda_n^*(\Omega_\varepsilon\cap R^c) \geq \lambda_n^*(E\cap R) + \lambda_n^*(E\cap R^c).\end{aligned}$$

We conclude that $\lambda_n^*(E) \geq \lambda_n^*(E\cap R) + \lambda_n^*(E\cap R^c)$.

If $\Omega_\varepsilon \cap \partial R \neq \emptyset$, we take the open set $\tilde{\Omega_\varepsilon} = \Omega_\varepsilon \setminus \partial R$. We have $\lambda_n^*(\Omega_\varepsilon) = \lambda_n^*(\tilde{\Omega_\varepsilon})$, this which ends the proof.

Let proving now that the measure λ_n is regular. Assume in the first time that A is a bounded measurable set. Let R be a closed rectangle which contains A and let $\varepsilon > 0$, the set $R \setminus A$ is measurable. There exists an open set $\Omega \supset (R \setminus A)$ such that $\lambda_n(\Omega) \leq \lambda_n(R \setminus A) + \varepsilon = \lambda_n(R) - \lambda_n(A) + \varepsilon$. The subset $K = R \cap \Omega^c$ is a compact and contained in A.

$$\lambda_n(R) = \lambda_n(R \cap \Omega^c) + \lambda_n(R \cap \Omega)) \leq \lambda_n(K) + \lambda_n(R) - \lambda_n(A) + \varepsilon$$

It results that $\lambda_n(A) \leq \lambda_n(K) + \varepsilon$.

If A is not bounded, we take an increasing sequence of compacts $(R_k)_k$ which cover $\mathbb{R}^n$. Then for each $k \in \mathbb{N}$, there exists a compact $K_k \subset R_k \cap A$ such that

$$\lambda_n(K_k) \geq \lambda_n(R_k \cap A) - \frac{1}{k+1}$$

It results that

$$\sup_{K \text{ compact} \subset A} \lambda_n(K) \geq \sup_k \lambda_n(K_k) \geq \lim_{k \to +\infty} (\lambda_n(R_k \cap A) - \frac{1}{k+1} = \lambda_n(A).$$

□

Theorem

A subset A of $\mathbb{R}^n$ is λ_n^*-measurable if and only if for any $\varepsilon > 0$ there exists an open set Ω containing A, such that $\lambda_n^*(\Omega \setminus A) \leq \varepsilon$.

Proof

Let A be a λ_n^*−measurable subset of $\mathbb{R}^n$. We assume that $\lambda_n^*(A) < +\infty$. Using Theorem (101), for $\varepsilon > 0$ there is an open set $\Omega \supset A$ such that $\lambda_n^*(\Omega) \le \lambda_n^*(A) + \varepsilon$. But

$$\lambda_n^*(\Omega) = \lambda_n^*(\Omega \cap A) + \lambda_n^*(\Omega \setminus A) = \lambda_n^*(A) + \lambda_n^*(\Omega \setminus A).$$

Thus, $\lambda_n^*(\Omega \setminus A) = \lambda_n^*(\Omega) - \lambda_n^*(A) \le \varepsilon$.

If $\lambda_n^*(A) = +\infty$. Let $A_j = A \cap B(0, j)$, for all $j \in \mathbb{N}$ there is an open set $\Omega_j \supset A_j$ and $\lambda_n^*(\Omega_j \setminus A_j) \leq \frac{\varepsilon}{2^j}$. The open set $\Omega = \bigcup_{j=1}^{+\infty} \Omega_j \supset A$ and

$$\lambda_n^*(\Omega \setminus A) = \lambda_n^*(\bigcup_{j=1}^{+\infty}(\Omega_j \setminus A_j)) \leq \varepsilon.$$

Conversely, let A be a subset of $\mathbb{R}^n$ such that for any $\varepsilon > 0$ there exists an open set Ω containing A, such that $\lambda_n^*(\Omega \setminus A) \leq \varepsilon$.

Let B be any subset of $\mathbb{R}^n$ such that $\lambda_n^*(B) < +\infty$. We need to show that $\lambda_n^*(B) \geq \lambda_n^*(B \cap A) + \lambda_n^*(B \cap A^c)$. By Corollary (103), there is a $\mathscr{G}_\delta$ set $\Omega \supset B$ such that $\lambda_n(\Omega) = \lambda_n^*(B)$. Then $\lambda_n^*(B) = \lambda_n(\Omega) = \lambda_n(\Omega \cap A) + \lambda_n(\Omega \cap A^c) \geq \lambda_n^*(B \cap A) + \lambda_n^*(B \cap A^c)$. The result is obtained by the same arguments as above if $\lambda_n^*(B) = +\infty$.

□

The following theorem gives the essential results of this section.

Theorem

For $A \in \mathscr{P}(\mathbb{R}^n)$, the following conditions are equivalent:
i) $A \in \mathscr{B}(\lambda_n^*)$.
ii) For every $\varepsilon > 0$, there exists an open set $\Omega \supset A$ such that $\lambda_n^*(\Omega \setminus A) < \varepsilon$.
iii) There exists a G_δ−set $G \supset A$ with $\lambda_n^*(G \setminus A) = 0$.
iv) For every $\varepsilon > 0$, there exists a closed set $F \subset A$ with $\lambda_n^*(A \setminus F) < \varepsilon$.
v) There exists an F_σ−set $F \subset A$ such that $\lambda_n^*(A \setminus F) = 0$.