

Math 106

Integral Calculus

Numerical Integration

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Numerical Integration

- 1 Rectangle Rules
- 2 Trapezoidal Rule
- 3 Simpson's Rule

Rectangle Rules

Rectangle Rules

Let $f : [a, b] \rightarrow \mathbb{R}$ with regular partition $P = \{x_0, \dots, x_n\}$ into n subintervals each of width $\Delta x = \frac{b-a}{n}$. Then the definite integral $\int_a^b f(x) dx$ can be approximated by

① the left rectangle rule

$$L_n = \sum_{k=1}^n f(x_{k-1}) \Delta x$$

② the right rectangle rule

$$R_n = \sum_{k=1}^n f(x_k) \Delta x$$

Trapezoidal Rule

Trapezoidal Rule

Let $f : [a, b] \rightarrow \mathbb{R}$ with regular partition $P = \{x_0, \dots, x_n\}$ into n subintervals each of width $\Delta x = \frac{b-a}{n}$. Then the definite integral $\int_a^b f(x) dx$ can be approximated by

$$\begin{aligned} T_n &= \frac{1}{2}(L_n + R_n) = \sum_{k=1}^n \frac{1}{2}[f(x_{k-1}) + f(x_k)]\Delta x \\ &= \frac{b-a}{2n}[f(x_0) + 2f(x_1) + 2f(x_2) + \dots + 2f(x_{n-1}) + f(x_n)] \end{aligned}$$

Trapezoidal Rule

Example

Approximate $\int_1^2 \frac{dx}{x}$ using trapezoidal rule with $n = 4$

Trapezoidal Rule

Example

Approximate $\int_1^2 \frac{dx}{x}$ using trapezoidal rule with $n = 4$

$$\Delta x = \frac{2-1}{4} = \frac{1}{4}$$

$$x_0 = 1, x_1 = \frac{5}{4}, x_2 = \frac{3}{2}, x_3 = \frac{7}{4}, x_4 = 2$$

$$\begin{aligned} T_n &= \frac{2-1}{2 \times 4} \left[f(1) + 2f\left(\frac{5}{4}\right) + 2f\left(\frac{3}{2}\right) + 2f\left(\frac{7}{4}\right) + f(2) \right] \\ &= \frac{1}{8} \left[1 + 2\left(\frac{4}{5}\right) + 2\left(\frac{2}{3}\right) + 2\left(\frac{4}{7}\right) + \frac{1}{2} \right] = \frac{1171}{1680} \approx 0.697 \end{aligned}$$

Exam Problem

Exam problem

Use the trapezoid rule with $n = 4$ to approximate $\int_1^4 \frac{1}{x} dx$

Simpson's Rule

Simpson's Rule

Let $f : [a, b] \rightarrow \mathbb{R}$ with regular partition $P = \{x_0, \dots, x_n\}$ into n subintervals **where n is even** each of width $\Delta x = \frac{b-a}{n}$. Then the definite integral $\int_a^b f(x) dx$ can be approximated by

$$S_n = \frac{b-a}{3n} [f(x_0) + 4f(x_1) + 2f(x_2) + \dots 4f(x_{n-1}) + f(x_n)]$$

Simpson's Rule

Example

Approximate $\int_1^2 \frac{dx}{x}$ using Simpson's rule with $n = 4$

Simpson's Rule

Example

Approximate $\int_1^2 \frac{dx}{x}$ using Simpson's rule with $n = 4$

$$\Delta x = \frac{2 - 1}{4} = \frac{1}{4}$$

$$x_0 = 1, x_1 = \frac{5}{4}, x_2 = \frac{3}{2}, x_3 = \frac{7}{4}, x_4 = 2$$

$$\begin{aligned} S_n &= \frac{2 - 1}{3 \times 4} \left[f(1) + 4f\left(\frac{5}{4}\right) + 2f\left(\frac{3}{2}\right) + 4f\left(\frac{7}{4}\right) + f(2) \right] \\ &= \frac{1}{12} \left[1 + 4\left(\frac{4}{5}\right) + 2\left(\frac{2}{3}\right) + 4\left(\frac{4}{7}\right) + \frac{1}{2} \right] = \frac{1747}{2520} \approx 0.6933 \end{aligned}$$