Math 106
Integral Calculus
Numerical Integration

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Numerical Integration

Rectangle Rules

Rectangle Rules

Let $f:[a,b]\to\mathbb{R}$ with regular partition $P=\{x_0,\dots,x_n\}$ into n subintervals each of width $\Delta x=\frac{b-a}{n}$. Then the definite integral $\int_a^b f(x)\,dx$ can be approximated by

1. the left rectangle rule

$$L_n=\sum_{k=1}^{n} f(x_{k-1})\Delta x$$

2. the right rectangle rule

$$R_n=\sum_{k=1}^{n} f(x_k)\Delta x$$

Trapezoidal Rule

Trapezoidal Rule

Let $f : [a, b] \to \mathbb{R}$ with regular partition $P = \{x_0, \dots, x_n\}$ into n subintervals each of width $\Delta x = \frac{b-a}{n}$. Then the definite integral $\int_a^b f(x)\,dx$ can be approximated by

$$T_n = \frac{1}{2}(L_n + R_n) = \sum_{k=1}^{n} \frac{1}{2}[f(x_{k-1}) + f(x_k)]\Delta x$$

$$= \frac{b-a}{2n}[f(x_0) + 2f(x_1) + 2f(x_2) + \dots 2f(x_{n-1}) + f(x_n)]$$

Trapezoidal Rule

Example

Approximate $\int_1^2 \frac{dx}{x}$ using trapezoidal rule with $n = 4$

Trapezoidal Rule

Example

Approximate $\int_1^2 \frac{dx}{x}$ using trapezoidal rule with $n = 4$

$$\Delta x = \frac{2-1}{4} = \frac{1}{4}$$

$$x_1 = 1, x_2 = \frac{5}{4}, x_3 = \frac{3}{2}, x_4 = \frac{7}{4}, x_5 = 2$$

$$T_n = \frac{2-1}{2 \times 4} \left[f(1) + 2f\left(\frac{5}{4}\right) + 2f\left(\frac{3}{2}\right) + 2f\left(\frac{7}{4}\right) + f(2) \right]$$

$$= \frac{1}{8} \left[1 + 2\left(\frac{4}{5}\right) + 2\left(\frac{2}{3}\right) + 2\left(\frac{4}{7}\right) + \frac{1}{2} \right] = \frac{1171}{1680} \approx 0.697$$

Error Estimation

Theorem

If f'' is continuous and $K \in \mathbb{R}^+$ such that $|f''(x)| \leq K$ for every $x \in [a, b]$, then the error estimates for trapezoidal rule of calculating $\int_a^b f(x)dx$ is

$$|E_t| = \left| \int_a^b f(x)dx - T_n \right| = \frac{K(b-a)^3}{12n^2}$$

Trapezoidal Rule

Example

Estimate the error of approximatiing $\int_1^2 \frac{dx}{x}$ using trapezoidal rule with $n = 4$

Trapezoidal Rule

Example

Estimate the error of approximatiing $\int_1^2 \frac{dx}{x}$ using trapezoidal rule with $n = 4$

Note that

$$f(x) = x^{-1}, \qquad f'(x) = -x^{-2}, \qquad f''(x) = 2x^{-3}$$

$$K = 2$$

$$|E_t| = \frac{2(2-1)^3}{12 \times 4^2} = \frac{1}{48}$$

Exam problem

Example

Use the trapezoid rule with $n = 4$ to approximate $\displaystyle\int_1^4 \frac{1}{x}dx$

Simpson's Rule

Simpson's Rule

Let $f : [a, b] \to \mathbb{R}$ with regular partition $P = \{x_0, \ldots, x_n\}$ into n subintervals where n is even each of width $\Delta x = \frac{b-a}{n}$. Then the definite integral $\int_a^b f(x)\,dx$ can be approximated by

$$S_n = \frac{b-a}{3n}[f(x_0) + 4f(x_1) + 2f(x_2) + \ldots 4f(x_{n-1}) + f(x_n)]$$

Simpson's Rule

Example

Approximate $\int_1^2 \frac{dx}{x}$ using Simpson's rule with $n = 4$

Example

Approximate $\int_1^2 \frac{dx}{x}$ using Simpson's rule with $n = 4$

$$\Delta x = \frac{2-1}{4} = \frac{1}{4}$$

$$x_1 = 1, x_2 = \frac{5}{4}, x_3 = \frac{3}{2}, x_4 = \frac{7}{4}, x_5 = 2$$

$$S_n = \frac{2-1}{3 \times 4}\left[f(1) + 4f\left(\frac{5}{4}\right) + 2f\left(\frac{3}{2}\right) + 4f\left(\frac{7}{4}\right) + f(2)\right]$$

$$= \frac{1}{12}\left[1 + 4\left(\frac{4}{5}\right) + 2\left(\frac{2}{3}\right) + 4\left(\frac{4}{7}\right) + \frac{1}{2}\right] = \frac{1747}{2520} \approx 0.6933$$

Error Estimation

Theorem

If $f^{(4)}$ *is continuous and* $K \in \mathbb{R}^+$ *such that* $|f^{(4)}(x)| \leq K$ *for every* $x \in [a, b]$, *then the error estimates for trapezoidal rule of calculating* $\int_a^b f(x)dx$ *is*

$$|E_t| = \left| \int_a^b f(x)dx - S_n \right| = \frac{K(b-a)^5}{180n^4}$$

Simpson's Rule

Example

Estimate the error of approximatiing $\int_1^2 \frac{dx}{x}$ using Simpson's rule with $n = 4$

Simpson's Rule

Example

Estimate the error of approximatiing $\int_1^2 \frac{dx}{x}$ using Simpson's rule with $n = 4$

Note that

$$f(x) = x^{-1}, \qquad f^{(4)}(x) = 24x^{-5}$$
$$K = 24$$

$$|E_S| = \frac{24(2-1)^5}{180 \times 4^4} = \frac{1}{1920}$$